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EFFECTS OF VISCOUS DISSIPATION ON THE FLUID FLOW THROUGH A THIN DOMAIN

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Abstract

In this work we investigate the flow and heat transfer inside a thin layer of fluid between two plane surfaces. The governing flow is described by the Stokes equation coupled with the heat equation involving the nonlinear viscous dissipation term. Using asymptotic analysis with respect to the layer thickness, a second-order effective model is proposed in the form of the explicit formulae for the velocity and temperature. Justification via error estimate is also discussed.

Keywords: non-isothermal flow, viscous dissipation, asymptotic approximation.

1 Introduction

We consider the situation in which two plane surfaces being in relative motion are separated by a thin layer of a viscous fluid. Motivated by the engineering applications (see e.g. [1]), we assume that the variations of the fluid temperature across the flow domain cannot be neglected. Thus, we start from the Stokes equation coupled with the heat equation involving the (nonlinear) viscous dissipation term. Our goal is to derive the effective equations governing the flow and heat transfer of this thin liquid film.

So far, numerous authors have addressed the isothermal case. However, the analytical treatments of the non-isothermal flow are rather sparse throughout the literature. We refer the reader to [2, 3, 4, 5]. The latter one is, in fact, a breakthrough paper because it takes into account the effects of the viscous dissipation term and, for the first time, provides the rigorous derivation of the (zero-order) non-isothermal Reynolds system. Following our idea proposed for isothermal setting (see [6, 7]), the aim of the present work is to propose a higher-order asymptotic model describing the macroscopic flow. To accomplish that, we employ multi-scale asymptotic expansion of the solution in powers of the small parameter ε, where $\Omega^\varepsilon = (0,1)^2 \times (0,\varepsilon)$ is a flow domain. Correctors in the asymptotic expansion are computed in a way that they have zero mean value and, as a result, the second-order effective model is obtained. Being in the form of the explicit formulae for the fluid velocity and temperature, it is particulary amenable for the numerical simulations. Furthermore, a rigorous justification of the formally derived asymptotic model is also discussed. In particular, the corresponding error estimates are provided in the case of the periodic boundary conditions. Let us mention that, in [8], a more general setting in which one of the surfaces is assumed to be rough has been rigorously addressed using similar approach. To conclude, we believe that the presented result could prove useful for improving the known engineering practice.

2 The problem

For a small parameter $\varepsilon > 0$, we introduce our simple three-dimensional thin domain as

$$\Omega^\varepsilon = \{(x_1, x_2, x_3) \in \mathbf{R}^3 \,:\, x' = (x_1, x_2) \in \mathcal{O}\,,\, 0 < x_3 < \varepsilon\}, \qquad \mathcal{O} = (0,1)^2\,. \tag{1}$$

As explained in the Introduction, we assume that the flow is governed by the following system:

$$-\mu\, \Delta \mathbf{u}^\varepsilon + \nabla p^\varepsilon = 0 \quad \text{in} \quad \Omega^\varepsilon\,, \tag{2}$$

$$\operatorname{div} \mathbf{u}^\varepsilon = 0 \quad \text{in} \quad \Omega^\varepsilon\,, \tag{3}$$

$$-\,\Delta \theta^\varepsilon = 2\mu\, |\mathbf{D}(\mathbf{u}^\varepsilon)|^2 \quad \text{in} \quad \Omega^\varepsilon\,. \tag{4}$$

The vector field $\mathbf{u}^\varepsilon$ denotes the fluid velocity, the pressure is given by the scalar field p^ε, whereas θ^ε represents the temperature of the fluid. The constant $\mu > 0$ corresponds to the viscosity of the fluid, while $\mathbf{D}(\mathbf{u}^\varepsilon) = \frac{1}{2}\left[\nabla \mathbf{u}^\varepsilon + (\nabla \mathbf{u}^\varepsilon)^T\right]$ is the symmetric part of the velocity gradient tensor. The following boundary conditions are imposed:

$$\mathbf{u}^\varepsilon = 0 \quad \text{for} \quad x_3 = \varepsilon\,, \tag{5}$$

$$\mathbf{u}^\varepsilon = \mathbf{s} \quad \text{for} \quad x_3 = 0\,, \tag{6}$$

$$\theta^\varepsilon = 0 \quad \text{for} \quad x_3 = 0,\, \varepsilon\,, \tag{7}$$

for constant velocity $\mathbf{s}$ of relative motion of two surfaces. Note that $\mathbf{s} \cdot \mathbf{k} = 0$ implying $u_3^\varepsilon|_{x_3=0} = 0$. Hereinafter, $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ denotes the standard Cartesian basis. In view of the applications we want to model, we assume that the flow is governed by the prescribed pressure drop $\frac{1}{\varepsilon^2}(q_0 - q_1)$ between sides $x_1 = 0$ and $x_1 = 1$. Our aim is to derive an effective law of higher order of accuracy describing the asymptotic behavior of the above described non-isothermal process.

3 Formal analysis

3.1 Stokes system

First, we deal with the system (2)-(3) together with boundary conditions (5)-(6). This part has been addressed earlier in [6] by the authors of the present paper (in more general setting). For reader's convenience, we recall the main results. We postulate the asymptotic expansion in the form

$$\mathbf{u}^\varepsilon = \mathbf{u}^0(x', y) + \varepsilon\, \mathbf{u}^1(x', y) + \varepsilon^2\, \mathbf{u}^2(x', y) + \cdots \quad , \tag{8}$$

$$p^\varepsilon = \frac{1}{\varepsilon^2}\, p^0(x') + \frac{1}{\varepsilon}\, p^1(x', y) + p^2(x', y) + \cdots\,, \quad y = \frac{x_3}{\varepsilon}\,. \tag{9}$$

In the sequel we use the following notation:

$$\nabla_{x'} \phi = \frac{\partial \phi}{\partial x_1}\, \mathbf{i} + \frac{\partial \phi}{\partial x_2}\, \mathbf{j}\,, \qquad \Delta_{x'} \mathbf{f} = \frac{\partial^2 \mathbf{f}}{\partial x_1^2} + \frac{\partial^2 \mathbf{f}}{\partial x_2^2}\,, \qquad \operatorname{div}_{x'} \mathbf{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2},$$

for a scalar function ϕ and a vector function $\mathbf{f} = f_1 \mathbf{i} + f_2 \mathbf{j} + f_3 \mathbf{k}$. Plugging the above expansions into the momentum equation (2) leads to

$$\begin{aligned} &\varepsilon^{-2} \left[-\mu\, \frac{\partial^2 \mathbf{u}^0}{\partial y^2} + \nabla_{x'} p^0 + \frac{\partial p^1}{\partial y}\, \mathbf{k}\right] + \varepsilon^{-1} \left[-\mu\, \frac{\partial^2 \mathbf{u}^1}{\partial y^2} + \nabla_{x'} p^1 + \frac{\partial p^2}{\partial y}\, \mathbf{k}\right] \\ &+ \left[-\mu\, \frac{\partial^2 \mathbf{u}^2}{\partial y^2} - \mu\, \Delta_{x'}\, \mathbf{u}^0 + \nabla_{x'} p^2 + \frac{\partial p^3}{\partial y}\, \mathbf{k}\right] + \cdots = 0\ . \end{aligned} \tag{10}$$

The main order term gives $p^1 = p^1(x')$ and

$$\mathbf{u}^0(x', y) = \frac{1}{2\mu} y\,(1-y)\;\mathbf{v}(x') + (1-y)\;\mathbf{s}\,, \quad \mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j}\,, \tag{11}$$

where

$$\mathbf{v} + \nabla_{x'}\, p^0 = 0\,. \tag{12}$$

Integrating (3) with respect to y, we obtain

$$\mathrm{div}_{x'}\,\mathbf{v} = 0 \ \text{ in } \ \mathcal{O}\,. \tag{13}$$

Now, we continue the computation by keeping the lower order terms in (10). Our aim is to build the correctors in a way that they have zero mean value $\int_0^1 \cdot\, dy$. For the first-order corrector, from the divergence equation, (11) and (13), we deduce $\mathbf{u}^1 = 0$, $p^1 = p^2 = 0$, while the second-order corrector $\mathbf{u}^2$ is given as the solution of the problem:

$$\begin{cases} -\mu\, \frac{\partial^2 \mathbf{u}^2}{\partial y^2} = \mu\, \Delta_{x'}\, \mathbf{u}^0 + \mathbf{A}(x')\,, \\ \mathbf{u}^2 = 0\,, \quad \text{for} \quad y = 0, 1\,, \\ \displaystyle\int_0^1 \mathbf{u}^2\; dy = 0\,. \end{cases} \tag{14}$$

Note that the additional term $\mathbf{A}(x')$ has to be introduced to the system since we required $\int_0^1 \mathbf{u}^2\; dy = 0$. Consequently, we get

$$\mathbf{u}^2(x', y) = \frac{1}{4\mu}\left(\frac{y^4}{6} - \frac{y^3}{3} + \frac{y^2}{5} - \frac{y}{30}\right)\Delta_{x'}\mathbf{v}\,. \tag{15}$$

Integrating the equation (10) with respect to y yields the Brinkman-type system for $(\mathbf{v}, p^0)$:

$$\mathbf{v} + \nabla_{x'}\, p^0 - \varepsilon^2\, \frac{1}{10}\, \Delta_{x'}\mathbf{v} = 0 \quad \text{in} \quad \mathcal{O}\,. \tag{16}$$

$$\mathrm{div}_{x'}\,\mathbf{v} = 0 \ \text{ in } \ \mathcal{O}\,. \tag{17}$$

Taking into account the prescribed boundary conditions, we can solve explicitly the above system to obtain:

$$\mathbf{v} = (q_0 - q_1)\left[1 + \frac{1 - e^{-\frac{\sqrt{10}}{\varepsilon}}}{e^{\frac{-2\sqrt{10}}{\varepsilon}} - 1}\left(e^{-\frac{\sqrt{10}\,x_2}{\varepsilon}} + e^{\frac{\sqrt{10}\,(x_2-1)}{\varepsilon}}\right)\right]\mathbf{i}\,, \tag{18}$$

$$p^0 = q_0 + (q_1 - q_0)x_1\,. \tag{19}$$

Observe that $\mathbf{v} = 0$ for $x_2 = 0, 1$.

3.2 Temperature distribution

Now we consider the heat equation

$$-\,\Delta\theta^\varepsilon = \frac{\mu}{2}\,|\nabla\mathbf{u}^\varepsilon + (\nabla\mathbf{u}^\varepsilon)^T|^2 \ \text{ in } \ \Omega^\varepsilon \tag{20}$$

endowed with the boundary conditions $\theta^\varepsilon(x', 0) = \theta^\varepsilon(x', \varepsilon) = 0$. We use the fact that

$$\nabla\mathbf{u}^\varepsilon + (\nabla\mathbf{u}^\varepsilon)^T = \begin{bmatrix} 2\,\frac{\partial u_1^\varepsilon}{\partial x_1} & \frac{\partial u_1^\varepsilon}{\partial x_2} + \frac{\partial u_2^\varepsilon}{\partial x_1} & \frac{\partial u_1^\varepsilon}{\partial x_3} + \frac{\partial u_3^\varepsilon}{\partial x_1} \\ \frac{\partial u_2^\varepsilon}{\partial x_1} + \frac{\partial u_1^\varepsilon}{\partial x_2} & 2\,\frac{\partial u_2^\varepsilon}{\partial x_2} & \frac{\partial u_2^\varepsilon}{\partial x_3} + \frac{\partial u_3^\varepsilon}{\partial x_2} \\ \frac{\partial u_3^\varepsilon}{\partial x_1} + \frac{\partial u_1^\varepsilon}{\partial x_3} & \frac{\partial u_3^\varepsilon}{\partial x_2} + \frac{\partial u_2^\varepsilon}{\partial x_3} & 2\,\frac{\partial u_3^\varepsilon}{\partial x_3} \end{bmatrix}. \tag{21}$$

Substituting the expansions

$$\mathbf{u}^\varepsilon = \mathbf{u}^0(x',y) + \varepsilon\, \mathbf{u}^1(x',y) + \varepsilon^2\, \mathbf{u}^2(x',y) + \cdots \quad , \tag{22}$$

$$\theta^\varepsilon = \theta^0(x',y) + \varepsilon\, \theta^1(x',y) + \varepsilon^2\, \theta^2(x',y) + \cdots \tag{23}$$

into (20), we get

$$\frac{1}{\varepsilon^2} \; : \quad -\frac{\partial^2 \theta^0}{\partial y^2} = \mu \left[\left(\frac{\partial u_1^0}{\partial y} \right)^2 + \left(\frac{\partial u_2^0}{\partial y} \right)^2 \right] . \tag{24}$$

Now we use the expression (11) for the zero-order velocity approximation to obtain

$$\begin{aligned} \theta^0(x',y) = \;& \frac{1}{4\mu} \left(-\frac{1}{3} y^4 + \frac{2}{3} y^3 - \frac{1}{2} y^2 + \frac{1}{6} y \right) |\mathbf{v}|^2 \\ & -\frac{\mu}{2} \left(y^2 - y \right) |\mathbf{s}|^2 + \left(-\frac{1}{3} y^3 + \frac{1}{2} y^2 - \frac{1}{6} y \right) (\mathbf{v} \cdot \mathbf{s}) . \end{aligned} \tag{25}$$

Analogously as above, we continue by computing the correctors in a way that $\int_0^1 \theta^1 dy = \int_0^1 \theta^2 dy = 0$. From (20) we obtain

$$\frac{1}{\varepsilon} \; : \quad \frac{\partial^2 \theta^1}{\partial y^2} = 0 \quad \Rightarrow \quad \theta^1(x',y) = 0 \, , \tag{26}$$

since $\theta^1(x',0) = \theta^1(x',1) = 0$. Next term is given by

$$1 \; : \; -\frac{\partial^2 \theta^2}{\partial y^2} = \Delta_{x'} \theta^0 + \frac{\mu}{2} \left(4 |\nabla_{x'} u^0|^2 - 2 \left(\frac{\partial u_1^0}{\partial x_2} - \frac{\partial u_2^0}{\partial x_1} \right)^2 + 4 \left(\frac{\partial u_3^1}{\partial y} \right)^2 \right) . \tag{27}$$

To meet the condition $\int_0^1 \theta^2 dy = 0$, we have to consider the following corrected problem for θ^2:

$$\begin{cases} -\frac{\partial^2 \theta^2}{\partial y^2} = \Delta_{x'} \theta^0 + \frac{\mu}{2} \left(4 |\nabla_{x'} \mathbf{u}^0|^2 - 2 \left(\frac{\partial u_1^0}{\partial x_2} - \frac{\partial u_2^0}{\partial x_1} \right)^2 + 4 \left(\frac{\partial u_3^1}{\partial y} \right)^2 \right) + D(x') \, , \\ \theta^2 = 0 \, , \quad \text{for} \quad y = 0, 1 \, , \\ \displaystyle\int_0^1 \theta^2 \, dy = 0 \, . \end{cases} \tag{28}$$

Using the expressions for $\mathbf{u}^0$, u_3^1 and θ^0, we finally obtain:

$$\begin{aligned} \theta^2(x',y) = \;& \frac{1}{2\mu} \left[\frac{1}{90} y^6 - \frac{1}{30} y^5 + \frac{1}{24} y^4 - \frac{1}{36} y^3 + \frac{1}{105} y^2 - \frac{1}{840} y \right] \left(|\nabla_{x'} \mathbf{v}|^2 + \mathbf{v} \cdot \Delta_{x'} \mathbf{v} \right) \\ & + \left[\frac{1}{60} y^5 - \frac{1}{24} y^4 + \frac{1}{36} y^3 - \frac{1}{360} y \right] (\Delta_{x'} \mathbf{v} \cdot \mathbf{s}) \\ & + \frac{1}{2\mu} \left[-\frac{1}{30} y^6 + \frac{1}{10} y^5 - \frac{1}{12} y^4 + \frac{3}{140} y^2 - \frac{1}{210} y \right] |\nabla_{x'} \mathbf{v}|^2 \\ & + \frac{1}{2\mu} \left[-\frac{1}{30} y^6 + \frac{1}{10} y^5 - \frac{1}{12} y^4 + \frac{3}{140} y^2 - \frac{1}{210} y \right] (\mathrm{div}_{x'} \mathbf{v})^2 \\ & + \frac{1}{4\mu} \left(\frac{1}{30} y^6 - \frac{1}{10} y^5 + \frac{1}{12} y^4 - \frac{3}{140} y^2 + \frac{1}{210} y \right) \left(\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} \right)^2 . \end{aligned} \tag{29}$$

4 Justification

Our formally derived effective model is given by the following expressions for the fluid velocity and temperature

$$\mathbf{u}^{\varepsilon}_{approx} = \mathbf{u}^0 + \varepsilon^2 \mathbf{u}^2 \,, \qquad \theta^{\varepsilon}_{approx} = \theta^0 + \varepsilon^2 \theta^2 \,. \tag{30}$$

It should be emphasized that all the terms in (30) have been computed in the explicit form, see Section 3. Now, we would like to discuss the justification via error estimates of such model. First, one should be aware that, in the case of the Dirichlet boundary conditions on $\partial\mathcal{O}$, we cannot hope to derive satisfactory error estimates in L^2 norm because of the boundary layer effects. Indeed, in (18), the function $e^{-\frac{\sqrt{10}\,x_2}{\varepsilon}} + e^{\frac{\sqrt{10}\,(x_2-1)}{\varepsilon}}$ appears being is of the boundary layer type. Thus, to avoid the boundary layers, we derive the error estimates in the case of the periodic boundary condition. More precisely, assuming that that all the quantities are periodic with respect to the horizontal variable x', an abstract result from [5] can be used to prove that the following estimates hold:

$$\left| \int_0^1 \mathbf{u}^{\varepsilon}\, dy \; - \; \frac{1}{12\mu}\mathbf{v} - \frac{1}{2}\mathbf{s} \right|_{L^2(\mathcal{O})} \leq C\varepsilon^3 \,, \tag{31}$$

$$\left| \varepsilon^2 \int_0^1 p^{\varepsilon}\, dy \; - \; p^0 \; \right|_{L^2(\mathcal{O})} \leq C\varepsilon^3 \,, \tag{32}$$

$$\left| \int_0^1 \theta^{\varepsilon}\, dy \; - \; \frac{1}{240\mu}\,|\mathbf{v}|^2 - \frac{\mu}{12}\,|\mathbf{s}|^2 \right|_{L^2(\mathcal{O})} \leq C\varepsilon \,. \tag{33}$$

The details can be found in [8].

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