

PROPERTIES AND CHARACTERISTICS OF THE FLUID FLOW MODEL WITH NON-EQUILIBRIUM THERMODYNAMIC PRESSURE

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Abstract

The paper presents a model of fluid flow with non-equilibrium pressure and the pressure evolution equation. The equation takes into account the dynamics of pressure changes caused by the difference between hydrodynamic and thermodynamic pressure as well as bulk viscosity effects. For the model of one-dimensional fluid flow, consisting of the conservation equations and the pressure evolution equation, the disturbances velocity was determined using the method of characteristics. Even though there occurs viscosity in the equations, the perturbation velocities are finite. The evolution equation for pressure makes it possible to understand pressure as an independent component of a vector of unknowns, similarly to density, velocity and internal energy. Thanks to this, the integration of the non-equilibrium pressure over time is performed as it is another unknown. The fluid flow solution is controlled by the presence of thermodynamic pressure in the form of an equilibrium equation of state.

Keywords: non-equilibrium pressure, relaxation time, evolution pressure equation, perturbation speed, Navier-Stokes equation.

1 Introduction

Fluid mechanics and heat transfer are highly developed fields of science, the foundations of which were laid in the 19th century. Their development in conjunction with the emergence of numerical methods has led to a situation in which both fields of science are used in many technical applications. Basically, it is assumed that conservation equations and various constitutive relationships that are at base of fluid mechanics have already adopted a classic form that does not require changes. This is true in many cases, however there are also issues where the classical approach does not give good results. One example is flow with phase transitions [1], where the difference between reality and calculation results is qualitative. In a shock wave and two-phase flows, the thermodynamic nonequilibrium is much higher than in homogeneous flows. So we are discussing flows with nonequilibrium, and yet one of the foundations on which fluid mechanics is built is local thermodynamic equilibrium. Although, as mentioned, the weaknesses of the equilibrium description of fluids are clearly visible in flows with phase transitions, further in our considerations we will refer to homogeneous flows, due to their simple description. In general case the following equations form the basis for the description of a fluid flow in the following form:

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \mathbf{v}) = 0, \quad (1)$$

$$\frac{\partial \varrho \mathbf{v}}{\partial t} + \operatorname{div}(\varrho \mathbf{v} \otimes \mathbf{v}) = -\operatorname{grad} p - \operatorname{div} \overset{\vee}{\mathbf{P}} + \varrho \mathbf{f}, \quad (2)$$

$$\frac{\partial \varrho u}{\partial t} + \operatorname{div}(\varrho u \mathbf{v}) = -\operatorname{div} \mathbf{q} - p \operatorname{div} \mathbf{v} - \overset{\vee}{\mathbf{P}} : \operatorname{div} \mathbf{v}, \quad (3)$$

where ϱ is density, $\mathbf{v}$ velocity, p hydrodynamic pressure, $\mathbf{f}$ mass force, u internal energy, $\mathbf{q}$ heat flux, and $\overset{\vee}{\mathbf{P}}$ is stress tensor. It is generally assumed that the pressure which is the trace of the stress tensor decomposes into thermodynamic pressure $\overset{\dagger}{p}$ and bulk viscous pressure $\overset{\vee}{p}$:

$$p = \overset{\dagger}{p} + \overset{\vee}{p} \quad (4)$$

The thermodynamic pressure $\overset{\dagger}{p}$ is the equilibrium pressure for fluid in ideal and stabilized conditions without flows and heat fluxes. The relationships between the thermodynamic parameters

of fluids are determined in very carefully conducted experiments, e.g. [2] and their results are equations of state. The results of such experiments are used in calculations where velocity and temperature gradients occur, sometimes of a very large value. Closing the system of equations (1)–(3) is possible after applying the law of Stokes, Newton and Fourier. According to the local equilibrium hypothesis, the instantaneous values of the thermodynamic parameters locally meet the equation of state. Assumption of local equilibrium allows to apply the state equation, for example in the form of thermodynamic pressure, which is a function of density and internal energy $\dot{p} = \dot{p}(\varrho, u)$. The problem described with balance equations and constitutive relationships is closed and can be solved. A characteristic feature of such system of equations with viscosity, thermal conductivity and second derivatives of velocity and temperature is the infinite speed of propagation of small disturbances.

In addition to the equilibrium treatment of the fluid, there are non-equilibrium approaches. In recent years, numerous papers on this topic have been published by Arima et al [5], Taniguchi et al [3], Arima and Ruggeri [4]. This approach is based on the principles of Extended Thermodynamics, which applies to phenomena far from equilibrium.

2 Flow model with non-equilibrium pressure

The purpose of the following considerations is to go beyond the rules of local equilibrium in a way that pressure essentially consist of thermodynamic and viscous pressure. The pressure, however will evolve in order to, after some sufficiently long time of zero flow gradient, reach to a point of equilibrium. Before the relationship based on Maxwell's proposal [6] is presented, we will simplify the balance equations. Let us consider a fluid motion in one direction where the heat flux $\mathbf{q}$ and the shear pressure $\overset{\circ}{P}$ are negligibly small. Additionally, let us assume that there are no mass forces $\mathbf{f}$. Such simplifying assumptions allow us to write the above system in the form of one-dimensional equations, as follows:

$$\frac{\partial \varrho}{\partial t} + \frac{\partial \varrho v}{\partial x} = 0, \quad (5)$$

$$\frac{\partial \varrho v}{\partial t} + \frac{\partial \varrho v^2}{\partial x} = -\frac{\partial p}{\partial x}, \quad (6)$$

$$\frac{\partial \varrho u}{\partial t} + \frac{\partial \varrho v u}{\partial x} = -p \frac{\partial v}{\partial x}. \quad (7)$$

Equations (5)–(7) describe the flow of a viscous fluid where the change in momentum occurs due to the pressure gradient and the fluid performs volumetric work. The above system of 3 equations contains 4 unknowns. Therefore, in order to close the system one additional pressure evolution differential equation will be introduced. In the proposed model, pressure evolves in time. So it does not only depend on the density and internal energy (thermodynamic pressure) and as a result of the effect of bulk viscosity and velocity gradient (viscous pressure), but also proportionally to the dynamics of pressure changes and relaxation time θ according to the following relationship:

$$\theta \frac{\partial p}{\partial t} + p = \dot{p}(\varrho, u) + \overset{\circ}{p}. \quad (8)$$

The above equation takes into account the inertia of the phenomenon. It means that the pressure p does not react immediately to changes in other flow parameters ϱ, v, u , but this happens with some delay. Time of equilibrium state θ is a measure of relaxation and is a characteristic property of fluid [7]. According to Cramer [8] the bulk viscosity, unlike the shear viscosity, can have various values. Given that the viscous volume pressure has the form:

$$\overset{\circ}{p} = -\xi \frac{\partial v}{\partial x} \quad (9)$$

where ξ is the bulk viscosity. The pressure evolution can be written as follows:

$$\frac{\partial p}{\partial t} = -\frac{1}{\theta} (p - \dot{p}) - \frac{\xi}{\theta} \frac{\partial v}{\partial x} \quad (10)$$

A property of the dynamics of pressure described in this way is its tendency to reach the thermodynamic pressure $\dot{p}$ for which the first term on the right side of the equation corresponds. A short relaxation time means a quick approach to the state of equilibrium, and a long relaxation time means that the pressure reacts poorly to changes in parameters of the flow. Pressure is also dependent on viscosity and velocity gradient. The negative velocity gradient causes fluid compression and additional pressure, and positive – fluid expansion and pressure drop.

Based on the equation (7) the internal energy in the volume changes as a result of the convective flow of energy and the work of pressure. With flow in the positive direction of spatial coordinate x , when $\partial v / \partial x < 0$, the work of thermodynamic pressure causes increase in internal energy. Under the same flow conditions, in the case of volumetric pressure work, things get complicated, because in general it can be both positive and negative. For $\partial v / \partial x < 0$ viscous volumetric pressure is positive, which causes an increase in internal energy liquid. As the velocity increases $\partial v / \partial x > 0$ the viscous pressure becomes negative.

The reference for our analyzes is the Navier-Stokes equation, where viscosity and second derivatives are present, resulting in infinite speed of small perturbations. In the context of the properties of the Navier-Stokes equation we will check what are characteristic velocities of the system of equations (5)–(8) and use the characteristic method to obtain these. To estimate the characteristics we will present the system of conservation equations and the pressure evolution equation in the non-conservative form:

$$\frac{\partial \varrho}{\partial t} + v \frac{\partial \varrho}{\partial x} + \varrho \frac{\partial v}{\partial x} = 0, \quad (11)$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{1}{\varrho} \frac{\partial p}{\partial x} = 0, \quad (12)$$

$$\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} + \frac{p}{\varrho} \frac{\partial v}{\partial x} = 0, \quad (13)$$

$$\frac{\partial p}{\partial t} + \frac{\xi}{\theta} \frac{\partial v}{\partial x} = -\frac{1}{\theta} (p - \dot{p}). \quad (14)$$

Taking the above into account, the system of equations from (11) do (14) can be written in a matrix form, as follows:

$$\frac{\partial \boldsymbol{\sigma}}{\partial t} + \mathbf{B}(\boldsymbol{\sigma}) \frac{\partial \boldsymbol{\sigma}}{\partial x} = \mathbf{c}(\boldsymbol{\sigma}) \quad (15)$$

where, if you choose the $\boldsymbol{\sigma}$ vector of unknowns as presented below, then the coefficient matrix $\mathbf{B}$ and the right side vector $\mathbf{c}$ are:

$$\boldsymbol{\sigma} = \begin{bmatrix} \varrho \\ v \\ u \\ p \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} v & \varrho & 0 & 0 \\ 0 & v & 0 & \frac{1}{\varrho} \\ 0 & \frac{p}{\varrho} & v & 0 \\ 0 & \frac{\xi}{\theta} & 0 & 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{(p - \dot{p})}{\theta} \end{bmatrix}.$$

Assuming $\mathbf{c} = 0$, the requirement for the existence of a non-zero solution to the equation (15) is that the matrix determinant $\mathbf{B} - \lambda \mathbf{I}$ meets the condition:

$$\det(\mathbf{B} - \lambda \mathbf{I}) = 0. \quad (16)$$

The equation (16) has four solutions:

$$\lambda_1 = v, \quad \lambda_2 = v, \quad \lambda_3 = v - a_\xi, \quad \lambda_4 = v + a.$$

where the small disturbances velocity is:

$$a = \sqrt{\frac{\xi}{\theta \varrho}}. \quad (17)$$

As one can see, there are four speeds of signal propagation, including two dependent ones on bulk viscosity and relaxation time. These signals go both ways flow directions. It is worth emphasizing that this phenomenon occurs at non-zero speed gradient.

Using the method of characteristics, let us consider a variant of the flow model fluid in which thermodynamic pressure is explicitly taken into account. In the momentum equation (12), we will substitute (14), and omit higher derivatives, so we get

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{1}{\varrho} \frac{\dot{p}(\varrho, u)}{\partial x} = 0. \quad (18)$$

Replacing equation (12) with the above equation, matrix $\mathbf{B}$ has the following form

$$\mathbf{B} = \begin{bmatrix} v & \varrho & 0 & 0 \\ \frac{1}{\varrho} \left(\frac{\partial \dot{p}}{\partial \varrho} \right)_u & v & 0 & \frac{1}{\varrho} \left(\frac{\partial \dot{p}}{\partial u} \right)_\varrho \\ 0 & \frac{p}{\varrho} & v & 0 \\ 0 & \frac{\xi}{\theta} & 0 & 0 \end{bmatrix}$$

In this case, the eigenvalue problem has four solutions

$$\lambda_1 = 0, \quad \lambda_2 = v, \quad \lambda_3 = v - a, \quad \lambda_4 = v + a$$

where the perturbation speed is

$$a = \sqrt{\left(\frac{\partial \dot{p}}{\partial u} \right)_\varrho \frac{p}{\varrho^2} + \left(\frac{\partial \dot{p}}{\partial \varrho} \right)_u}. \quad (19)$$

The sound velocity given by the equation (19) depends on the thermodynamic parameters of density and internal energy, because thermodynamic pressure is a function of ϱ and u . But in formula (19) there is hydrodynamic pressure, which includes a viscous volumetric pressure $\dot{p}$ dependent on volumetric viscosity coefficient and velocity derivative $\partial v / \partial x$.

3 Solution of the initial problem with the evolutionary equation for non- equilibrium pressure

A one-dimensional fluid flow model consists of a balance equation mass (11), momentum (12), energy (13) and volume pressure evolution equation (14). Despite viscosity it is characterized by a finite speed of propagation signals. This is a feature that makes this model qualitatively different than the models described by the Navier-Stokes equation. System of equations (11) – (14) is hyperbolic, while system of equations with the Navier-Stokes equation would be parabolic. This qualitative difference results not only in distinct solutions, but also in other methods of searching for the solutions. A system of balance equations with pressure in the form of an evolutionary equation (10) allows to solve the flow problem in a completely different way. Pressure loses its unique feature, which was independence from time, and now occurs in the equation where time is explicitly present.

The initial boundary problem describing the flow consists of a system of equations (11) – (14) written in matrix form as (15), initial conditions

$$\boldsymbol{\sigma}(t_0) = \boldsymbol{\sigma}^0. \quad (20)$$

and appropriate boundary conditions. In a case where every unknown exists in the form of a derivative, the integration of each equation after time can be applied. Integrating the equation (15) in a range of time t^{n+1} and t^n gives:

$$\int_{t^n}^{t^{n+1}} \frac{\partial \boldsymbol{\sigma}}{\partial t} dt = - \int_{t^n}^{t^{n+1}} \mathbf{B}(\boldsymbol{\sigma}) \frac{\partial \boldsymbol{\sigma}}{\partial x} dt + \int_{t^n}^{t^{n+1}} \mathbf{c}(\boldsymbol{\sigma}) dt. \quad (21)$$

Without going into the details of numerical methods, and using the simplest differential scheme, one may obtain an equation in which there is an explicit vector of unknowns at time $n + 1 = t^{n+1}$:

$$\sigma^{n+1} = \sigma^n - B \frac{\sigma_{i+1} - \sigma_i}{\Delta x} \Delta t + c \Delta t \quad (22)$$

where index i is a point in calculation space. Thanks to the proposed character of pressure equation with time derivative, all unknowns: ϱ , v , u and p can be solved by integrating the relevant equations over time. The solution depends on the equation of state because the vector $\mathbf{c}$ contains thermodynamic pressure. If there are no thermodynamic forces acting on the fluid, it tends to equilibrium.

4 Summary

The paper proposes a flow model in which the pressure is described by differential equation. The equation for the evolution of pressure includes thermodynamic pressure, but also the relaxation time and time derivative. Equations describing the fluid in this form are of a different type. They are not parabolic and become hyperbolic and the speed of the signals becomes finite. In mathematical terms, pressure becomes a parameter similar to other flow quantities like density, velocity and internal energy because it involves a differential equation.

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