

ON GRAVITY CURRENTS IN POWER-LAW CROSS-SECTION CHANNELS: THE EFFECT OF VEGETATION

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Abstract

We consider the propagation of gravity currents (GC), through an array of vertical obstacles, called vegetation, along channels of power-law cross-section (CS) shapes. We formulate a model and solve it numerically. Special attention is given to the similarity long-term behaviour of the current, for which the analytical solutions are available and compared to the numerical data.

Keywords: Gravity currents, shallow-water, vegetation.

1 Introduction

Gravity currents (GCs) occur whenever fluid of one density spreads primarily horizontally into fluid of different density. The classic case is a flow spreading at the bottom of a rectangular channel. But in nature and industrial constructions, most channels have a non-rectangular geometry (rivers, valleys, human-made systems, etc.), therefore it is important to investigate the influence of the different channel geometries on the motion of the gravity currents. Many examples of such currents were discussed in the previous studies of Simpson [4], Marino, *et al.* [2], Monaghan, *et al.* [3] and Zemach & Ungarish, [9].

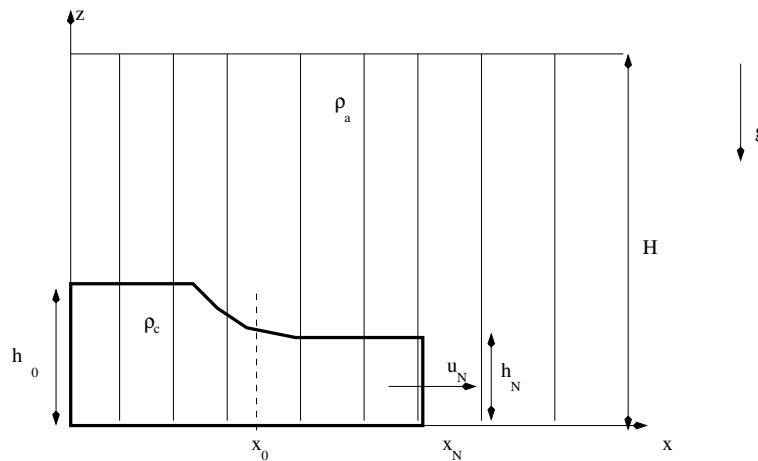


Figure 1: Schematic description of the current released from a lock of length x_0 and height h_0 in the channel of height H with power-law cross-section: side view

Recently, the influence of the vegetation on the gravity currents was investigated. Tanino, *et al.* [5] represented lock-exchange experiments; Hatcher and Hogg [1] modelled the propagation of a turbulent GC through an array of obstacles; the theoretical analysis for the one- and two-layer configurations was done by Testik and Ungarish [6]. However, all these works consider the flows in rectangular channels.

The effect of the geometry of the channel was also discussed in different studies. Zemach and Ungarish [9] developed a one-layer shallow-water model for the high-Reynolds-number propagation; Zemach [8] discussed a one-layer shallow-water (SW) model for particle-driven gravity currents in

non-rectangular CS channels for turbulent and laminar flows [8]. However, all these works were limited to the GC propagation in the containers with no vegetation.

In the present work we develop a shallow-water model for the cross-sections of power-law shapes, that takes into a consideration the effect of vegetation. The model describes the behaviour of the gravity current of the constant volume. We consider the systems of the Boussinesq (Bq) type, where the density differences between the current and the ambient are relatively small. The effect of the vegetation is modelled as a drag term. Following Hatcher, *et al.* [1], Tanino, *et al.* [5] and

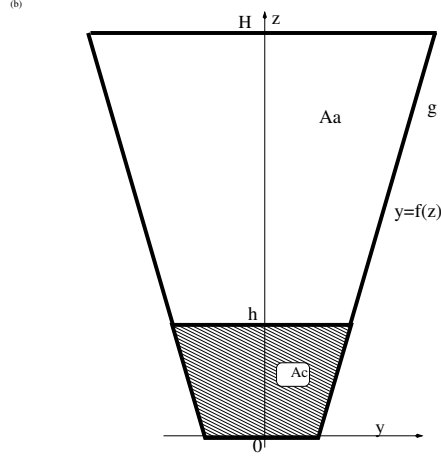


Figure 2: Cross-section of channel. Here $f(z)$ is the width of the channel. In the analysis A_a denotes the area occupied by the ambient fluid, A_c is the area occupied by the current, and $A_T = A_c + A_a$ is the total area.

Ungarish [7] the fluid spreads through the obstacles is affected by the internal drag force, that is proportional to the speed of the current at some power λ ($|u|^\lambda$). The effective drag per unit volume of fluid is proportional to the density of the fluid and to the speed at power $\lambda \geq 1$. The system of equations that describes the motion, includes continuity and momentum balances. We focus on the flow, where the drag is dominant and turbulent and viscous stresses are negligible.

The structure of the paper is as follows. In Section 2, the shallow-water model is formulated. In Section 3 similarity analytical solutions are presented and compared with the numerical data. Finally, in Section 4 some concluding remarks are given.

2 Formulation

Consider a gravity current created by release of a fluid of density ρ_c and kinematic viscosity ν into an ambient fluid of constant density ρ_a and propagating in a horizontal channel of a uniform in the x -direction cross-section shape in container of height H , which is filled by a vertical-rod-like vegetation.

The system under consideration is sketched in Figs. 1-2: the bottom and the top of the half-infinite channel are at $z = 0$ and $z = H$ and its side-walls are given by $y = -f(z)$ and $y = f(z)$ functions, where $f(z)$ is assumed to be continuous and positive and to be of the power-law form $f(z) = bz^\gamma$ (where b and γ are constants). Gravity acts in the $-z$ direction. The current propagates in the positive x -direction. The driving force is the reduced gravity of the current which is defined by

$$g' = (\rho_c - \rho_a)g/\rho_a, \quad (1)$$

where g is the acceleration due to gravity.

A $\{x, y, z\}$ Cartesian coordinate system with corresponding $\{u, v, w\}$ velocity components is employed. The fluids are assumed to be separated by a sharp, non-entraining, interface which is flat (horizontal) in the y direction.

One-layer SW model is used. For the ambient fluid domain, we assume $\rho = \rho_a$, $u = v = w = 0$.

The dimensional variables are used, unless stated otherwise. The position of the interface (the thickness of the current) is $h(x, t)$ and its velocity, averaged over the area of the current, is $u(x, t)$. Initially, at $t = 0$, $h = h_0$ and $u = 0$. We assume a shallow current with $h_0/x_0 \ll 1$. The Reynolds number of the horizontal flow, $Re = h_N u_N / \nu$, where the subscript N denotes value associated with the nose of the current, is assumed to be large.

Let A_c and A_T denote the total cross section of the current and the total of the channel:

$$\begin{cases} A_c = A(h) = \int_0^h f(z) dz = \frac{bh^\gamma}{\gamma+1}, \\ A_T = \int_0^H f(z) dz = \frac{bH^\gamma}{\gamma+1}. \end{cases} \quad (2)$$

It is convenient to use the dimensionless variables defined as follows (here the dimensional variables are denoted by an asterisk):

$$\{x^*, z^*, h^*, H^*, t^*, u^*\} = \{x_0 x, h_0 z, h_0 h, h_0 H, T t, U u\}, \quad (3)$$

where $U = (h_0 g')^{1/2}$ and $T = x_0 / U$. The y -direction variables are scaled with the width of the interface in the lock, $f(h_0)$.

2.1 The governing equations

We start with the balances between the pressure gradient and the drag in each one of the current and the ambient layers:

$$\frac{dp_a}{dx} - \rho_a \tilde{c}_1 u_a |u_a|^{\lambda_1-1} = 0; \quad (4)$$

$$\frac{dp_a}{dx} - \Delta \rho \cdot g \frac{dh}{dx} - \rho_c \tilde{c}_2 u |u|^{\lambda_2-1} = 0 \quad (5)$$

Here $\tilde{c}_1$ and $\tilde{c}_2$ are the drag coefficients in the ambient and the current layers. We assume that the internal drag force of each layer is proportional to the speed $|u|$: at some power λ_1 for the layer of the current ($|u|^{\lambda_1}$) and at some power λ_2 for the ambient layer ($|u|^{\lambda_2}$).

Next, we eliminate the pressure term and using

$$u_a A_a = u A \quad (6)$$

we obtain

$$0 = -g' \frac{dh}{dx} - \tilde{c}_2 u |u|^{\lambda_2-1} \left[1 + \frac{\rho_a}{\rho_c} \frac{\tilde{c}_1}{\tilde{c}_2} \left(\frac{A}{A_a} \right)^{\lambda_1} |u|^{\lambda_1-\lambda_2} \right]. \quad (7)$$

Next, we assume that the coefficient $\tilde{c}$ is constant. In respect with $\tilde{c}_1 = \tilde{c}_2 = C$, $\lambda_1 = \lambda_2 = \lambda$, taking into account the one-layer model (for which $\frac{h}{H} \rightarrow 0$ and $\frac{A}{A_a} \rightarrow 0$) and using the dimensionless parameters (3), the continuity and the momentum equations can be written in the dimensionless form by

$$\begin{cases} h_t + u h_x + \frac{h}{\gamma+1} u_x = 0; \\ -h_x - u |u|^{\lambda-1} = 0. \end{cases} \quad (8)$$

The domain of the current is $0 \leq x \leq x_N(t)$, where $x_N(t)$ denotes the distance of a propagation of the front (nose) of the current. We use the following stretched coordinate: $y = \frac{x}{x_N(t)}$ that keeps a flow at the fixed domain $y \in [0, 1]$. Thus,

$$\begin{cases} \left(\frac{\partial}{\partial t} \right)_x = \left(\frac{\partial}{\partial t} \right)_y - y \frac{\dot{x}_N}{x_N} \left(\frac{\partial}{\partial y} \right)_x, \\ \left(\frac{\partial}{\partial x} \right)_t = \frac{1}{x_N} \left(\frac{\partial}{\partial y} \right)_x. \end{cases} \quad (9)$$

Now, the system of equations (8) can be written in terms of (y, t) by

$$\begin{cases} h_t + \frac{1}{x_N} \left[(u - y \dot{x}_N) h_y + \frac{h}{\gamma+1} u_y \right] = 0, \\ -h_y \frac{1}{x_N} - u |u|^{\lambda-1} = 0. \end{cases} \quad (10)$$

Finally, since from (10), $u = \left(\frac{-h_y}{x_N}\right)^{\frac{1}{\lambda}}$, the system of equations (10) can be reduced to the singular second-order parabolic equation for the height h :

$$h_t + y \frac{\dot{x}_N}{x_N} P - \frac{1}{x_N^{m+1}} \left[P^{m+1} + \frac{h}{\gamma+1} m P^{m-1} h_{yy} \right] = 0, \quad (11)$$

where

$$m = \frac{1}{\lambda}, \quad P = -h_y. \quad (12)$$

We assume that the volume of the current is $V = q = \text{const}$.

For the solution, we use an explicit finite difference method to obtain numerically first the height $h(x, t)$ of the current and then the velocity $u(x, t)$ and the distance of propagation $x_N(t)$ of the current.

3 Similarity solutions

In this section, we present an analytical solution of a self-similar type. The self-similar solution of the SW equations is relevant for long times and only for the rectangular and power-low cross-sections. We assume a solution of a self-similar form by:

$$x_N = Kt^\beta; \quad h = t^\delta C \mathcal{H}(y); \quad u = \dot{x}_N \mathcal{U}(y), \quad (13)$$

where the upper dot means a time derivative; $K, \beta, \delta = \text{const}$; $\mathcal{U}, K, \mathcal{H} - \text{dimensionless}$ and $C = K^{\lambda+1} \beta^\lambda$.

The boundary conditions at the nose ($y = 1$) are:

$$\mathcal{U}(1) = 1, \quad \mathcal{H}(1) = 0. \quad (14)$$

Substitution of the self-similar variables (13) into the system of equations (10) yields:

$$\begin{cases} \frac{\delta}{\beta} \mathcal{H}(y) + (\mathcal{U} - y) \mathcal{H}'(y) + \frac{1}{\gamma+1} \mathcal{H}(y) \mathcal{U}'(y) = 0, \\ (\mathcal{U}(y))^\lambda = -\mathcal{H}'(y), \end{cases} \quad (15)$$

and

$$\delta = -\lambda + \beta(\lambda + 1). \quad (16)$$

Consider the volume:

$$V = \int_0^{x_N} A_c(h) dx = \int_0^{x_N} \frac{h^{\gamma+1}}{\gamma+1} dx = \frac{Kt^{\beta+\delta(\gamma+1)}}{\gamma+1} \int_0^1 [\mathcal{H}(y)]^{\gamma+1} dy = q. \quad (17)$$

We obtain:

$$\beta + \delta(\gamma + 1) = 0, \quad KJ = q(\gamma + 1), \quad J = \int_0^1 [\mathcal{H}(y)]^{\gamma+1} dy. \quad (18)$$

Using the volume balances (18) and (16), we obtain the distance of propagation $x_N(t) = Kt^\beta$, where

$$\beta = \frac{\lambda(\gamma + 1)}{1 + (\lambda + 1)(\gamma + 1)}. \quad (19)$$

We summarize:

$$\beta + \delta(\gamma + 1) = 0, \quad \delta = -\frac{\beta}{\gamma + 1}, \quad \beta = \frac{\lambda(\gamma + 1)}{1 + (\lambda + 1)(\gamma + 1)}. \quad (20)$$

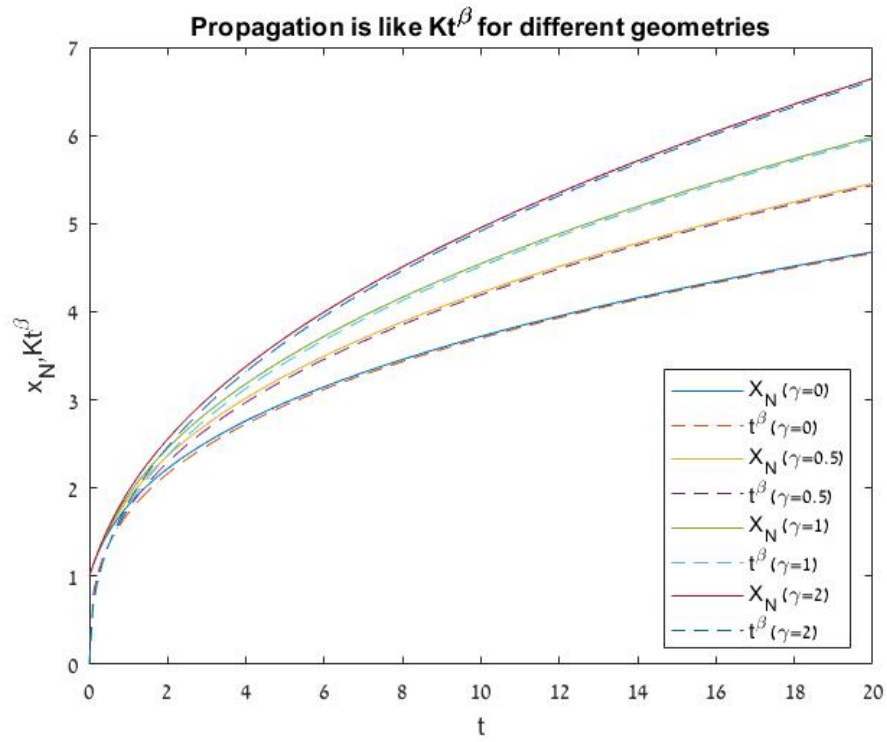


Figure 3: An influence of γ on a specific drag with $\lambda = 1$: propagation of the current is like Kt^β . x_N - solid lines, Kt^β - dashed lines

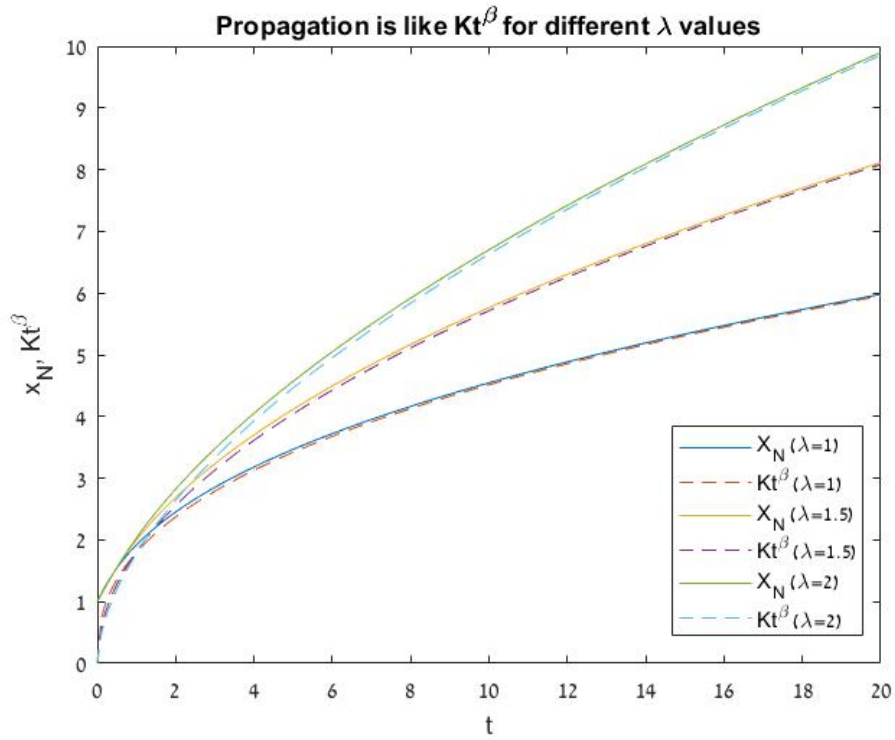


Figure 4: An influence of $\lambda = 1, 1.5, 2$ in containers with $f(z) = z$: propagation of the current is like Kt^β . x_N - solid lines, Kt^β - dashed lines

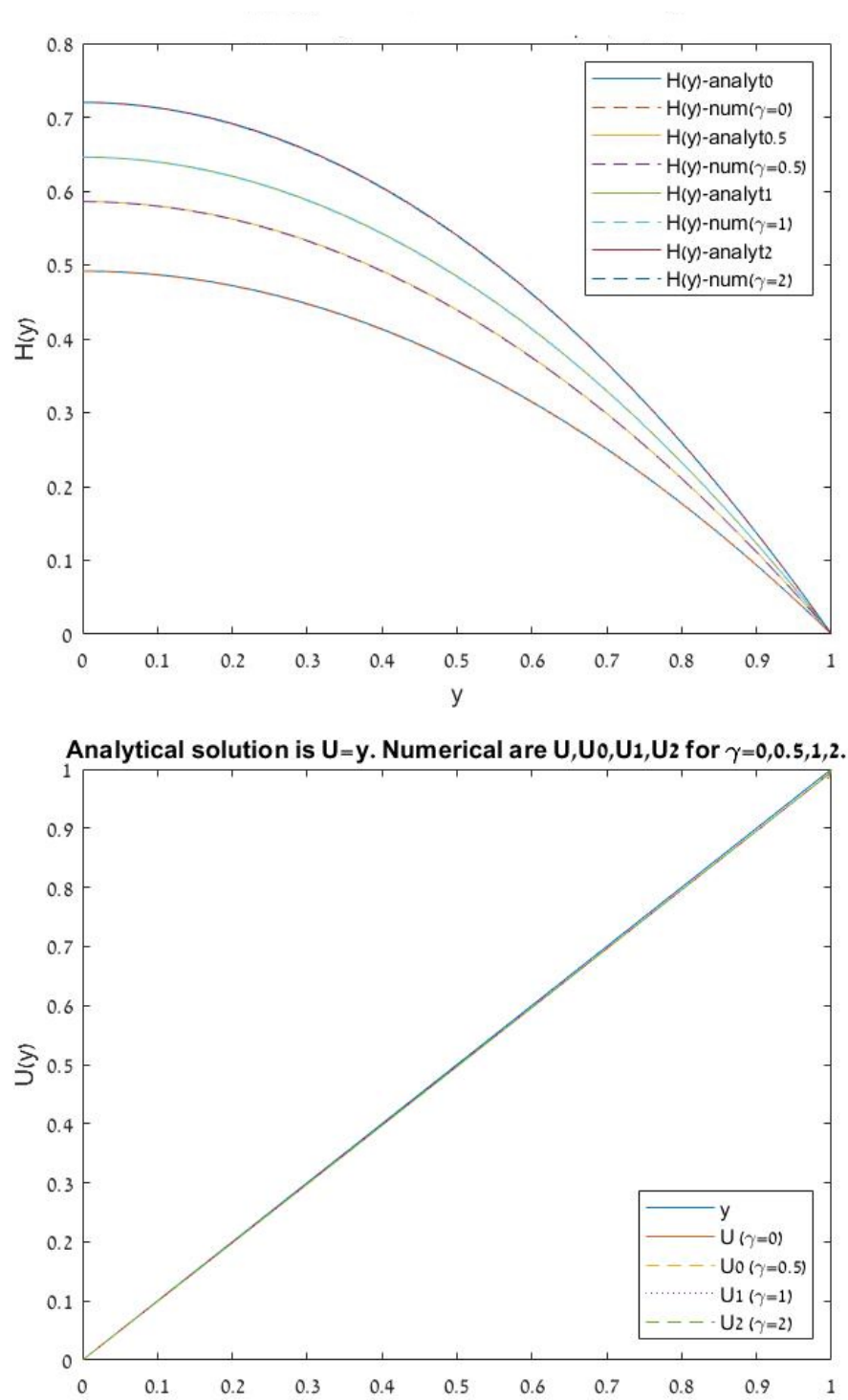


Figure 5: $H(y)$ and $U(y)$ profiles for $\lambda = 1$ and various cross sections $f(z) = z^\gamma$ ($\gamma = 0, 0.5, 1, 2$)

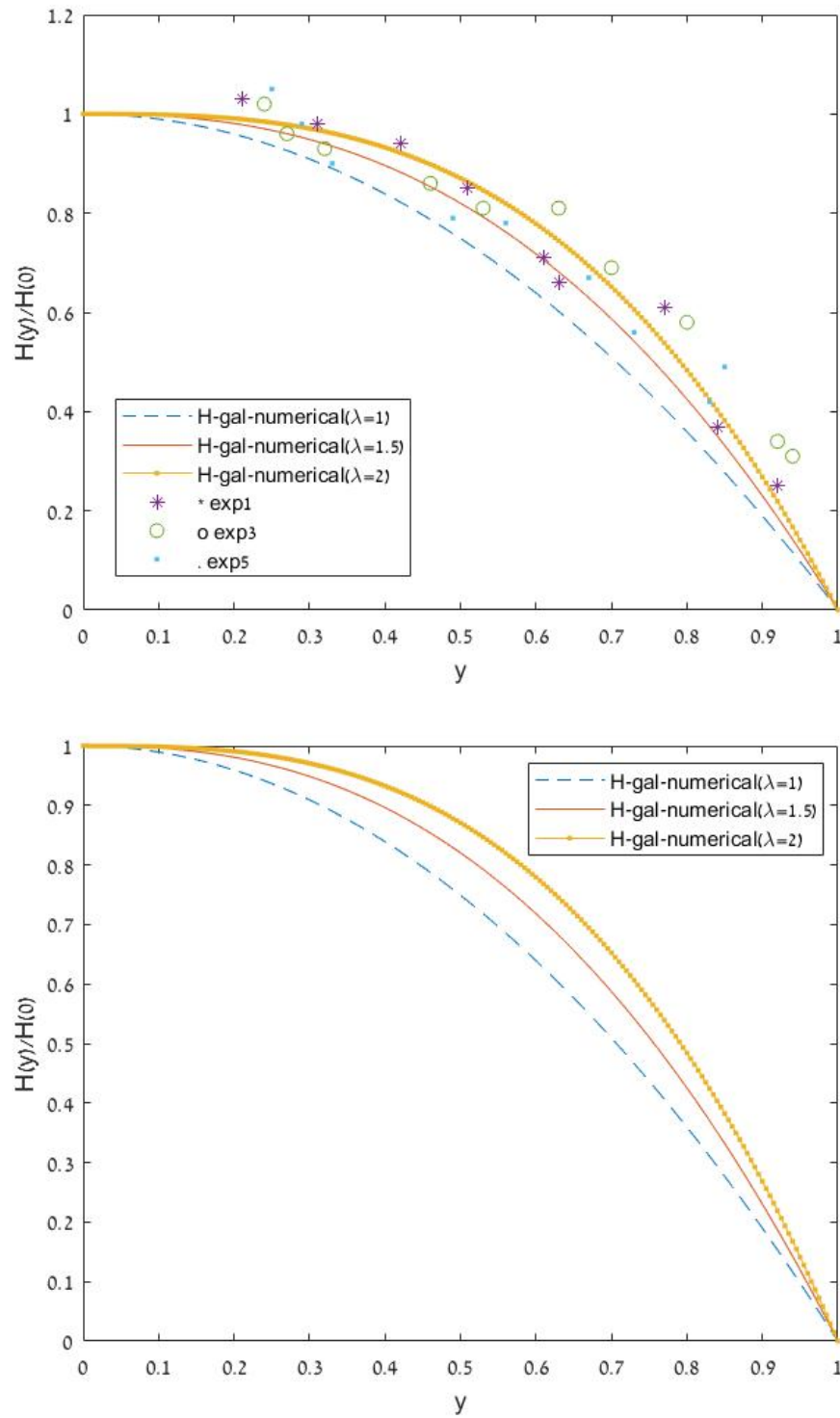


Figure 6: Profiles of $\mathcal{H}(y)/\mathcal{H}(0)$ for $\lambda \in \{1, 1.5, 2\}$. (a) for the rectangular CS ($\gamma = 0$). The data include also the experimental data of Hatcher [1]. (b) for the triangular CS ($\gamma = 1$).

For $(1 - y) \ll 1$ (i.e. x close to x_N), the solution of (15) can be approximated by :

$$\mathcal{H} = (1 - y) - \frac{\lambda}{2} \left(\frac{\gamma + 1}{\gamma + 2} \right) \left(1 - \frac{\delta}{\beta} \right) (1 - y)^2 + \dots, \quad (21)$$

$$\mathcal{U} = 1 - \left(\frac{\gamma + 1}{\gamma + 2} \right) \left(1 - \frac{\delta}{\beta} \right) (1 - y) + \dots. \quad (22)$$

Substitution of (20) into (21) and using the boundary conditions (14) finally provides the similarity solutions:

$$\mathcal{U} = y; \quad \mathcal{H}(y) = \frac{1}{\lambda + 1} (1 - y^{1+\lambda}). \quad (23)$$

It can be verified by the substitution that the solution (23) satisfies the system of equations (15).

Fig.4 shows the profiles of the distance of propagation $x_N(t)$ for $f(z) = z^\gamma$ ($\gamma = 1, 2$) and for the different values of λ ($\lambda = 1, 1.5, 2$) obtained for the analytical solution (23) and compared to the numerical data obtained from (10). The current propagates as $x_N \sim t^\beta$. We note a very good agreement between the numerical and the analytical results. The current moves more slowly for $\lambda = 1$ than for $\lambda > 1$ for all examined cross-section shapes. We speculate that such behaviour was expected because of the effect of an inertial/turbulent drag force created by the obstacles. A similar influence on the motion was observed also in the rectangular channels. For a fixed value of λ , a distance of propagation increases when the value of γ increases (Fig.3) and a slowest propagation is observed in the rectangular channel. Fig.5 illustrates the numerical and analytical solutions for various power-law CS ($\gamma = 0, 0.5, 1, 2$). The function $\mathcal{U}(y)$ is linear for all power-law CS, and the analytical solution $\mathcal{H}(y)$ is supported by the numerical solution. At Fig.6 we show the behaviour of $\mathcal{H}(y)$ for specific cross-section shapes and different λ values ($\lambda \in \{1, 1.5, 2\}$). We compare the rectangular case results ($\gamma = 0$) to the experimental data of Hatcher [1]. It is relevant to compare these results for $\lambda = 1.5$ and $\lambda = 2$. We note, that $\mathcal{H}(y)$ is affected by the value of λ similarly for all considered cross sections.

4 Conclusions.

In the present work, we developed a model that simulates the presence of a vegetation in the channels of different cross-section shapes. We concentrate on the containers of a power-law CS given by a function $f(z) = z^\gamma$. The inertial drag is proportional to $|u|^\lambda$, where λ is constant in the range ($1 \leq \lambda \leq 2$). For all considered cases, a faster propagation was observed as λ increases.

Our work also elucidates similarities and differences between the propagation of the current in the power-law cross-section and in the classical rectangular case. Quantitatively, the slowest propagation is always achieved in rectangular containers for any value of λ .

Unfortunately, there are no experimental or Navier-Stokes-simulation data for comparison with the model. The interpretation and implications of this performance requires further investigation. We hope that this paper will motivate for such experiments and Navier-Stokes simulations in the near future.

References

- [1] Hatcher, L., Hogg, A. J., Woods, A. W.: The effects of drag on turbulent gravity currents. *Journal of Fluid Mechanics*, 416: (2000) pp.297–314.
- [2] Marino, B. M., & Thomas, L. P.: Dam-break release of a gravity current in a power-law channel section. *Journal of Physics: Conference Series*, 296: (2011), pp.012008.
- [3] Monaghan, J.J., Meriaux, C.A., Huppert, H.E., and Mansour, J.: Particulate gravity currents along V-shaped valleys. *J. Fluid Mech.* 631:(2011) pp. 419-438
- [4] Simpson, J.: Gravity currents in the environment and the laboratory. *Cambridge University* (1997) Press.

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- [5] Tanino, Y., Nepf, H. M., and Kulis, P. S.: Gravity currents in aquatic canopies. *Water Resour. Res.* 41, 12:(2005) pp. W12402–1–9
 - [6] Testik, F. Y., Ungarish, M.: On the self-similar propagation of gravity currents through an array of emergent vegetation-like obstacles. *Physics of Fluids*, 28(5):(2016) pp.056605
 - [7] M.Ungarish: Gravity Currents and Intrusions: Analysis and Prediction, *CRC Press* (2020).
 - [8] Zemach, T.: Particle-driven gravity currents in non-rectangular cross section channels. *Physics of Fluids*, 27(10):(2015), pp.103303.
 - [9] Zemach, T., Ungarish, M.: Gravity currents in non-rectangular cross section channels: Analytical and numerical solutions of the one-layer shallow water model for high-Reynolds-number propagation. *Physics of Fluids*, 25(2):(2013) pp.026601.