

MOTION OF FLUIDS IN THE MOVING DOMAIN

Š. Nečasová

Institute of Mathematics of the Czech Academy of Sciences, Žitná 25, 11567 Praha 1, Czech Republic

Abstract

It is a survey paper where the problem of the existence of weak solutions of compressible barotropic solutions in a moving bounded domain is studied.

Keywords: compressible fluid, moving domain, weak solutions

1 Introduction

Modelling of the problems of fluid flow inside a moving domain deserve a lot of interest as they appear in many practical applications. These types of problems can also be seen as a preparation step for research of fluid-structure interaction problems. It is therefore important to understand what difficulties associated with the motion of the domain arise in the mathematical theory of fluid flows.

Research of the global existence of weak solutions of compressible Navier-Stokes system dates back to the nineties. The groundbreaking result of the existence of the global weak solutions to the compressible barotropic Navier-Stokes system on a fixed domain was proved by P. L. Lions [15]. Later, by E. Feireisl and his collaborators [10] such result was extended to more physically relevant state equations and also to the full thermodynamical system : the Navier-Stokes-Fourier system [3].

Let us stress that the boundary conditions on the interfaces play a crucial role in the problems involving the motion of solid objects in fluids and occur frequently in various applications of continuum fluid dynamics.

Besides the commonly used *no-slip* condition, where the velocity of the fluid coincides with that of the adjacent solid body, various slip-like conditions have been proposed to handle the situations in which the no-slip scenario fails to produce a correct description of the fluid boundary behavior. For *viscous* fluids, Navier proposed the boundary conditions in the form

$$[\mathbb{S}\mathbf{n}]_{\tan} + \kappa [\mathbf{u} - \mathbf{V}]_{\tan} |_{\Gamma_\tau} = 0, \quad \kappa \geq 0, \quad (1.1)$$

where $\mathbb{S}$ is the viscous stress tensor, κ represents a “friction” coefficient, $\mathbf{u}$ and $\mathbf{V}$ denote the fluid and solid body velocities, respectively, and Γ_τ is the position of the interface at a time τ , with the outer normal vector $\mathbf{n}$. If $\kappa = 0$, we obtain the *complete slip* while the asymptotic limit $\kappa \rightarrow \infty$ gives rise to the standard no-slip boundary conditions.

We will consider the standard *Navier-Stokes system*:

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0, \quad (1.2)$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \varrho \mathbf{f}, \quad (1.3)$$

where ϱ is the density, $p = p(\varrho)$ the (barotropic) pressure, $\mathbf{f}$ a given external force, and $\mathbb{S}$ is determined by the standard *Newton rheological law*

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{I} \right) + \eta \operatorname{div}_x \mathbf{u} \mathbb{I}, \quad \mu > 0, \quad \eta \geq 0. \quad (1.4)$$

The boundary of the domain Ω_t occupied by the fluid is described by means of a *given* velocity field $\mathbf{V}(t, \mathbf{x})$, where $t \geq 0$ and $\mathbf{x} \in R^3$. More specifically, assuming $\mathbf{V}$ is regular, we solve the associated system of differential equations

$$\frac{d}{dt} \mathbf{X}(t, \mathbf{x}) = \mathbf{V}(t, \mathbf{X}(t, \mathbf{x})), \quad t > 0, \quad \mathbf{X}(0, \mathbf{x}) = \mathbf{x}, \quad (1.5)$$

and set

$\Omega_\tau = \mathbf{X}(\tau, \Omega_0)$, where $\Omega_0 \subset \mathbb{R}^3$ is a given domain, $\Gamma_\tau = \partial\Omega_\tau$, and $Q_\tau = \{(t, x) \mid t \in (0, \tau), x \in \Omega_\tau\}$.

In addition to (1.1), we assume that the boundary Γ_τ is impermeable, meaning

$$(\mathbf{u} - \mathbf{V}) \cdot \mathbf{n}|_{\Gamma_\tau} = 0 \text{ for any } \tau \geq 0. \quad (1.6)$$

Finally, the problem (1.1 - 1.6) is supplemented by the initial conditions

$$\varrho(0, \cdot) = \varrho_0, \quad (\varrho \mathbf{u})(0, \cdot) = (\varrho \mathbf{u})_0 \text{ in } \Omega_0. \quad (1.7)$$

In this survey paper we will focus on the simplest case of barotropic situation, see [7] where we will consider the Navier type of boundary conditions. Our main goal is to show *existence* of global-in-time *weak solutions* to problem (1.1 - 1.7) for any finite energy initial data.

We introduce the following *three-level approximation scheme*:

1. *penalization*

$$\frac{1}{\varepsilon} \int_0^T \int_{\Gamma_t} (\mathbf{u} - \mathbf{V}) \cdot \mathbf{n} \, \varphi \cdot \mathbf{n} \, dS_x \, dt, \quad \varepsilon > 0 \text{ small}. \quad (1.8)$$

Remark: Such an approach was proposed by Stokes and Carey [17]. We will apply the penalty approach to slip conditions. In the time-dependent geometries, the penalization can be applied in the *interior* of a fixed reference domain, however, the resulting limit system consists of *two* fluids separated by impermeable boundary and coupled through the tangential components of normal stresses. In such a way, an extra term is produced acting on the fluid by its “complementary” part outside Ω_τ . In order to eliminate these extra stresses we need another parameters.

2. *variable* shear viscosity coefficient $\mu = \mu_\omega$, where μ_ω remains strictly positive in the fluid domain Q_T but vanishes in the solid domain Q_T^c as $\omega \rightarrow 0$.
3. the *artificial pressure* (Similarly to the existence theory developed in [4])

$$p_\delta(\varrho) = p(\varrho) + \delta \varrho^\beta, \quad \beta \geq 2, \quad \delta > 0,$$

in the momentum equation (1.3).

The main ideas of the proof of the existence of solutions:

1. Keeping $\varepsilon, \delta, \omega > 0$ fixed, we solve the modified problem in a (bounded) reference domain $B \subset \mathbb{R}^3$ chosen in such a way that

$$\overline{\Omega}_\tau \subset B \text{ for any } \tau \geq 0.$$

We adapt the existence theory for the compressible Navier-Stokes system with variable viscosity coefficients developed in [5].

2. We take the initial density ϱ_0 vanishing outside Ω_0 , and letting $\varepsilon \rightarrow 0$ for fixed $\delta, \omega > 0$ we obtain a “two-fluid” system, where the density vanishes in the solid part $((0, T) \times B) \setminus Q_T$ of the reference domain. (see Fundamental lemma 3.1.)
3. Letting the viscosity vanish in the solid part, we perform the limit $\omega \rightarrow 0$, where the extra stresses disappear in the limit system.
4. The desired conclusion results from the final limit process $\delta \rightarrow 0$ and also from the construction of test functions.

1.1 Bibliographical remarks

In the context of compressible fluid flow systems in time-varying domains, we first address the work [9], where the existence of weak solutions of the barotropic compressible Navier-Stokes systems with nonslip boundary conditions $\mathbf{u} = \mathbf{V}$ on Γ_τ was proven. After that by Feireisl et al. [7] the existence of weak solutions of the barotropic compressible Navier-Stokes systems has been treated on time-dependent domains in the case of slip-boundary conditions (as prescribed in (1.2)–(1.7)) was shown. Their approach is based on the penalization of the boundary behavior, viscosity and pressure in the weak formulation. We will give more details in the next sections. Later on, the existence of weak solutions to the full *Navier-Stokes-Fourier* (N-S-F) systems in the time-dependent domain have been analyzed by Kreml et al. [13, 12]. Recently, the authors in [16] studied the existence of a weak solution to a nonlinear fluid-structure interaction model with heat exchange where the shell is governed by linear thermoelasticity equations and encompasses a time-dependent domain that is filled with a fluid governed by the full N-S-F system. In this regard, we also mention the work [11], where the authors analyze a system governing the interaction between two compressible mutually non-interacting fluids and a shell of Koiter-type that actually leads to a time-dependent domain filled by the fluids. In this sense we see, that the existence in the case of moving domain is kind of preliminary work for the study of fluid-structure interaction problem.

Last but not least, we mention that the existence of weak solutions to the 3-D Navier-Stokes-Fourier-Poisson in the time-varying domain has been explored in [1] with nonhomogeneous temperature on the boundary where the heat flux does not necessarily satisfy the conservative boundary conditions. The question of weak-strong uniqueness and existence of the strong solution in L^2 case was shown in [14]. There are also a few results of the problem of singular limits in a moving domain, see [6, 8].

2 Formulation of the problem

In the weak formulation, it is convenient that the equation of continuity (1.2) holds in the whole physical space R^3 provided the density ϱ was extended to be zero outside the fluid domain, specifically

$$\int_{\Omega_\tau} \varrho \varphi(\tau, \cdot) \, dx - \int_{\Omega_0} \varrho_0 \varphi(0, \cdot) \, dx = \int_0^\tau \int_{\Omega_t} (\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi) \, dx \, dt \quad (2.1)$$

for any $\tau \in [0, T]$ and any test function $\varphi \in C_c^\infty([0, T] \times R^3)$. Moreover, equation (1.2) is also satisfied in the sense of renormalized solutions introduced by DiPerna and Lions [2]:

$$\begin{aligned} \int_{\Omega_\tau} b(\varrho) \varphi(\tau, \cdot) \, dx - \int_{\Omega_0} b(\varrho_0) \varphi(0, \cdot) \, dx = \\ \int_0^\tau \int_{\Omega_t} (b(\varrho) \partial_t \varphi + b(\varrho) \mathbf{u} \cdot \nabla_x \varphi + (b(\varrho) - b'(\varrho) \varrho) \operatorname{div}_x \mathbf{u} \varphi) \, dx \, dt \end{aligned} \quad (2.2)$$

for any $\tau \in [0, T]$, any $\varphi \in C_c^\infty([0, T] \times R^3)$, and any $b \in C^1[0, \infty)$, $b(0) = 0$, $b'(r) = 0$ for large r . Of course, we suppose that $\varrho \geq 0$ a.a. in $(0, T) \times R^3$.

Similarly, the momentum equation (1.3) is replaced by a family of integral identities

$$\begin{aligned} \int_{\Omega_\tau} \varrho \mathbf{u} \cdot \boldsymbol{\varphi}(\tau, \cdot) \, dx - \int_{\Omega_0} (\varrho \mathbf{u})_0 \cdot \boldsymbol{\varphi}(0, \cdot) \, dx \\ = \int_0^\tau \int_{\Omega_t} (\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho [\mathbf{u} \otimes \mathbf{u}] : \nabla_x \boldsymbol{\varphi} + p(\varrho) \operatorname{div}_x \boldsymbol{\varphi} - \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \boldsymbol{\varphi} + \varrho \mathbf{f} \cdot \boldsymbol{\varphi}) \, dx \, dt \end{aligned} \quad (2.3)$$

for any $\tau \in [0, T]$ and any test function $\boldsymbol{\varphi} \in C_c^\infty([0, T] \times R^3; R^3)$ satisfying

$$\boldsymbol{\varphi} \cdot \mathbf{n}|_{\Gamma_\tau} = 0 \text{ for any } \tau \in [0, T]. \quad (2.4)$$

Finally, the impermeability condition (1.6) is satisfied in the sense of traces, specifically,

$$\mathbf{u} \in L^2(0, T; W^{1,2}(R^3; R^3)) \text{ and } (\mathbf{u} - \mathbf{V}) \cdot \mathbf{n}(\tau, \cdot)|_{\Gamma_\tau} = 0 \text{ for a.a. } \tau \in [0, T]. \quad (2.5)$$

At this stage, we are ready to state the main result of the present paper:

Theorem 2.1 [Theorem 2.1, [7]] Let $\Omega_0 \subset R^3$ be a bounded domain of class $C^{2+\nu}$, and let $\mathbf{V} \in C^1([0, T]; C_c^3(R^3; R^3))$ be given. Assume that the pressure $p \in C[0, \infty) \cap C^1(0, \infty)$ satisfies

$$p(0) = 0, \quad p'(\varrho) > 0 \text{ for any } \varrho > 0, \quad \lim_{\varrho \rightarrow \infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty > 0 \text{ for a certain } \gamma > 3/2.$$

Let the initial data satisfy

$$\varrho_0 \in L^\gamma(R^3), \quad \varrho_0 \geq 0, \quad \varrho_0 \not\equiv 0, \quad \varrho_0|_{R^3 \setminus \Omega_0} = 0, \quad (\varrho \mathbf{u})_0 = 0 \text{ a.a. on the set } \{\varrho_0 = 0\},$$

$$\int_{\Omega_0} \frac{1}{\varrho_0} |(\varrho \mathbf{u})_0|^2 dx < \infty.$$

Then the problem (1.1 - 1.7) admits a weak solution on any time interval $(0, T)$ in the sense specified through (2.1 - 2.5).

3 Main ideas of the proof of Theorem 2.1

For the sake of simplicity, we restrict ourselves to the case $\kappa = 0$, $\eta = 0$, and $\mathbf{f} = 0$. As we shall see, the main ideas of the proof presented below require only straightforward modifications to accommodate the general case.

3.1 Approximate problem - weak formulation

Choosing $R > 0$ such that

$$\mathbf{V}|_{[0, T] \times \{|\mathbf{x}| > R\}} = 0, \quad \bar{\Omega}_0 \subset \{|\mathbf{x}| < R\} \quad (3.1)$$

we take the reference domain $B = \{|\mathbf{x}| < 2R\}$.

Next, the shear viscosity coefficient $\mu = \mu_\omega(t, \mathbf{x})$ is taken such that

$$\mu_\omega \in C_c^\infty([0, T] \times R^3), \quad 0 < \underline{\mu}_\omega \leq \mu_\omega(t, x) \leq \mu \text{ in } [0, T] \times B, \quad \mu_\omega(\tau, \cdot)|_{\Omega_\tau} = \mu \text{ for any } \tau \in [0, T]. \quad (3.2)$$

Finally, we define modified initial data so that

$$\varrho_0 = \varrho_{0, \delta}, \quad \varrho_{0, \delta} \geq 0, \quad \varrho_{0, \delta} \not\equiv 0, \quad \varrho_{0, \delta}|_{R^3 \setminus \Omega_0} = 0, \quad \int_B \left(\varrho_{0, \delta}^\gamma + \delta \varrho_{0, \delta}^\beta \right) dx \leq c, \quad (3.3)$$

$$(\varrho \mathbf{u})_0 = (\varrho \mathbf{u})_{0, \delta}, \quad (\varrho \mathbf{u})_{0, \delta} = 0 \text{ a.a. on the set } \{\varrho_{0, \delta} = 0\}, \quad \int_{\Omega_0} \frac{1}{\varrho_{0, \delta}} |(\varrho \mathbf{u})_{0, \delta}|^2 dx \leq c. \quad (3.4)$$

The weak formulation of the *approximate problem* reads as follows:

$$\int_B \varrho \varphi(\tau, \cdot) dx - \int_B \varrho_0 \varphi(0, \cdot) dx = \int_0^\tau \int_B (\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi) dx dt \quad (3.5)$$

for any $\tau \in [0, T]$ and any test function $\varphi \in C_c^\infty([0, T] \times R^3)$;

$$\begin{aligned} & \int_B \varrho \mathbf{u} \cdot \varphi(\tau, \cdot) dx - \int_B (\varrho \mathbf{u})_0 \cdot \varphi(0, \cdot) dx \\ &= \int_0^\tau \int_B (\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho [\mathbf{u} \otimes \mathbf{u}] : \nabla_x \varphi + p(\varrho) \operatorname{div}_x \varphi + \delta \varrho^\beta \operatorname{div}_x \varphi) dx dt \\ & \quad - \int_0^\tau \int_B \left(\mu_\omega \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbf{I} \right) : \nabla_x \varphi \right) dx dt \\ & \quad + \frac{1}{\varepsilon} \int_0^\tau \int_{\Gamma_t} ((\mathbf{V} - \mathbf{u}) \cdot \mathbf{n} \varphi \cdot \mathbf{n}) dS_x dt \end{aligned} \quad (3.6)$$

for any $\tau \in [0, T]$ and any test function $\varphi \in C_c^\infty([0, T] \times B; R^3)$, where $\mathbf{u} \in L^2(0, T; W_0^{1,2}(B; R^3))$, meaning $\mathbf{u}$ satisfies the no-slip boundary condition

$$\mathbf{u}|_{\partial B} = 0 \text{ in the sense of traces.} \quad (3.7)$$

Here, ε , δ , and ω are positive parameters. The choice of the no-slip boundary condition (3.7) is not essential.

The *existence* of global-in-time solutions to the penalized problem can be shown by means of the method developed in [5] to handle the nonconstant viscosity coefficients. Indeed, for $\varepsilon > 0$ fixed, the extra penalty term in (3.6) can be treated as a “compact” perturbation. In addition, solutions can be constructed satisfying the *energy inequality*

$$\begin{aligned} & \int_B \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) + \frac{\delta}{\beta-1} \varrho^\beta \right) (\tau, \cdot) \, dx + \frac{1}{2} \int_0^\tau \int_B \mu_\omega \left| \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{I} \right|^2 \, dx \, dt \\ & + \frac{1}{\varepsilon} \int_0^\tau \int_{\Gamma_t} [(\mathbf{u} - \mathbf{V}) \cdot \mathbf{n}] \mathbf{u} \cdot \mathbf{n} \, dS_x \, dt \leq \int_B \left(\frac{1}{2\varrho_{0,\delta}} |(\varrho \mathbf{u})_{0,\delta}|^2 + P(\varrho_{0,\delta}) + \frac{\delta}{\beta-1} \varrho_{0,\delta}^\beta \right) \, dx, \end{aligned} \quad (3.8)$$

where

$$P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \, dz.$$

Note that the quantity on the right-hand side of (3.8) representing the total energy of the system is finite because of (3.3), (3.4).

In addition, since $\beta \geq 2$, the density is square integrable and we may use the regularization technique of DiPerna and Lions [2] to deduce the renormalized version of (3.5), namely

$$\int_B b(\varrho) \varphi(\tau, \cdot) \, dx - \int_B b(\varrho_0) \varphi(0, \cdot) \, dx = \int_0^\tau \int_B (b(\varrho) \partial_t \varphi + b(\varrho) \mathbf{u} \cdot \nabla_x \varphi + (b(\varrho) - b'(\varrho) \varrho) \operatorname{div}_x \mathbf{u} \varphi) \, dx \, dt \quad (3.9)$$

for any φ and b as in (2.2).

3.2 Limits $\varepsilon \rightarrow 0$, $\omega \rightarrow 0$, and $\delta \rightarrow 0$

In this section, we perform successively the singular limits $\varepsilon \rightarrow 0$, $\omega \rightarrow 0$, and $\delta \rightarrow 0$.

3.2.1 Penalization limit: $\varepsilon \rightarrow 0$

Keeping the parameters δ , ω fixed, our goal is to let $\varepsilon \rightarrow 0$ in (3.5), (3.6). Let $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon\}$ be the corresponding weak solution of the perturbed problem constructed in the previous section. Then we can reach the boundary conditions on Γ_τ

$$(\mathbf{u} - \mathbf{V}) \cdot \mathbf{n}(\tau, \cdot)|_{\Gamma_\tau} = 0 \text{ for a.a. } \tau \in [0, T]. \quad (3.10)$$

Passing to the limit in (3.5) we obtain

$$\int_B \varrho \varphi(\tau, \cdot) \, dx - \int_B \varrho_{0,\delta} \varphi(0, \cdot) \, dx = \int_0^\tau \int_B (\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi) \, dx \, dt \quad (3.11)$$

for any $\tau \in [0, T]$ and any test function $\varphi \in C_c^\infty([0, T] \times R^3)$.

The limit in the momentum equation (3.6) is more delicate. Since the surfaces Γ_τ are determined *a priori*, we can use the technique based on the so-called Bogovskii operator to deduce the uniform bounds $\int \int_{\mathcal{K}} (p(\varrho) \varrho^\nu + \delta \varrho^{\beta+\nu}) \, dx \, dt \leq c(\mathcal{K})$ for a certain $\nu > 0$ for any compact $\mathcal{K} \subset [0, T] \times \bar{B}$ such that $\mathcal{K} \cap \left(\bigcup_{\tau \in [0, T]} (\{\tau\} \times \Gamma_\tau) \right) = \emptyset$. Since we have at hand only the *local estimates* on the pressure, we have to restrict ourselves to the class of test functions

$$\varphi \in C^1([0, T]; W_0^{1,\infty}(B; R^3)), \operatorname{supp}[\operatorname{div}_x \varphi(\tau, \cdot)] \cap \Gamma_\tau = \emptyset, \varphi \cdot \mathbf{n}|_{\Gamma_t} = 0 \text{ for all } \tau \in [0, T]. \quad (3.12)$$

Passing to the limit in (3.6), we obtain

$$\begin{aligned} & \int_B \varrho \mathbf{u} \cdot \boldsymbol{\varphi}(\tau, \cdot) \, dx - \int_B (\varrho \mathbf{u})_{0,\delta} \cdot \boldsymbol{\varphi}(0, \cdot) \, dx \\ &= \int_0^\tau \int_B \left(\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho [\mathbf{u} \otimes \mathbf{u}] : \nabla_x \boldsymbol{\varphi} + \overline{p(\varrho)} \operatorname{div}_x \boldsymbol{\varphi} + \delta \overline{\varrho^\beta} \operatorname{div}_x \boldsymbol{\varphi} \right) \, dx \, dt \\ & \quad - \int_0^\tau \int_B \left(\mu_\omega \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{I} \right) : \nabla_x \boldsymbol{\varphi} \right) \, dx \, dt \end{aligned} \quad (3.13)$$

for any test function $\boldsymbol{\varphi}$ as (3.12). Note that the requirement of smoothness of $\boldsymbol{\varphi}$ postulated in (3.6) can be easily relaxed by means of a density argument.

From the method developed in [5] it can be shown pointwise (a.a.) convergence of the sequence $\{\varrho_\varepsilon\}_{\varepsilon>0}$

$$\varrho_\varepsilon \rightarrow \varrho \text{ a.a. in } (0, T) \times B. \quad (3.14)$$

In accordance with (3.14), the momentum equation (3.13) reads

$$\begin{aligned} & \int_B \varrho \mathbf{u} \cdot \boldsymbol{\varphi}(\tau, \cdot) \, dx - \int_B (\varrho \mathbf{u})_{0,\delta} \cdot \boldsymbol{\varphi}(0, \cdot) \, dx \\ &= \int_0^\tau \int_B \left(\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho [\mathbf{u} \otimes \mathbf{u}] : \nabla_x \boldsymbol{\varphi} + p(\varrho) \operatorname{div}_x \boldsymbol{\varphi} + \delta \varrho^\beta \operatorname{div}_x \boldsymbol{\varphi} \right) \, dx \, dt \\ & \quad - \int_0^\tau \int_B \left(\mu_\omega \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{I} \right) : \nabla_x \boldsymbol{\varphi} \right) \, dx \, dt \end{aligned} \quad (3.15)$$

for any test function $\boldsymbol{\varphi}$ as in (3.12). In addition, as already observed, the limit solution $\{\varrho, \mathbf{u}\}$ satisfies also the renormalized equation (3.9).

Fundamental lemma

Our next goal is to use the specific choice of the initial data $\varrho_{0,\delta}$ to get rid of the density-dependent terms in (3.15) supported by the “solid” part $((0, T) \times B) \setminus Q_T$. To this end, we show the following result, rather obvious for regular solutions but a bit more delicate in the weak framework, that may be of independent interest.

Lemma 3.1 *Let $\varrho \in L^\infty(0, T; L^2(B))$, $\varrho \geq 0$, $\mathbf{u} \in L^2(0, T; W_0^{1,2}(B; \mathbb{R}^3))$ be a weak solution of the equation of continuity, specifically,*

$$\int_B \left(\varrho(\tau, \cdot) \boldsymbol{\varphi}(\tau, \cdot) - \varrho_0 \boldsymbol{\varphi}(0, \cdot) \right) dx = \int_0^\tau \int_B \left(\varrho \partial_t \boldsymbol{\varphi} + \varrho \mathbf{u} \cdot \nabla_x \boldsymbol{\varphi} \right) \, dx \, dt \quad (3.16)$$

for any $\tau \in [0, T]$ and any test function $\boldsymbol{\varphi} \in C_c^1([0, T] \times \mathbb{R}^3)$.

In addition, assume that

$$(\mathbf{u} - \mathbf{V})(\tau, \cdot) \cdot \mathbf{n}|_{\Gamma_\tau} = 0 \text{ for a.a. } \tau \in (0, T), \quad (3.17)$$

and that

$$\varrho_0 \in L^2(\mathbb{R}^3), \quad \varrho_0 \geq 0, \quad \varrho_0|_{B \setminus \Omega_0} = 0.$$

Then

$$\varrho(\tau, \cdot)|_{B \setminus \Omega_\tau} = 0 \text{ for any } \tau \in [0, T].$$

Thus, by virtue of Lemma 3.1, the momentum equation (3.15) reduces to

$$\int_{\Omega_\tau} \varrho \mathbf{u} \cdot \boldsymbol{\varphi}(\tau, \cdot) \, dx - \int_{\Omega_0} (\varrho \mathbf{u})_{0,\delta} \cdot \boldsymbol{\varphi}(0, \cdot) \, dx \quad (3.18)$$

$$\begin{aligned}
&= \int_0^\tau \int_{\Omega_t} (\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho [\mathbf{u} \otimes \mathbf{u}] : \nabla_x \boldsymbol{\varphi} + p(\varrho) \operatorname{div}_x \boldsymbol{\varphi} + \delta \varrho^\beta \operatorname{div}_x \boldsymbol{\varphi}) \, dx \, dt \\
&\quad - \int_0^\tau \int_{\Omega_t} \left(\mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{I} \right) : \nabla_x \boldsymbol{\varphi} \right) \, dx \, dt \\
&\quad - \int_0^\tau \int_{B \setminus \Omega_t} \mu_\omega \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{I} \right) : \nabla_x \boldsymbol{\varphi} \, dx \, dt
\end{aligned}$$

for any test function $\boldsymbol{\varphi}$ as in (3.12). We remark that it was exactly this step when we needed the extra pressure term $\delta \varrho^\beta$ ensuring the density ϱ to be square integrable.

3.2.2 Vanishing viscosity limit: $\omega \rightarrow 0$

In order to get rid of the last integral in (3.18), we take the viscosity coefficient

$$\mu_\omega = \begin{cases} \mu = \text{const} > 0 & \text{in } Q_T, \\ \mu_\omega \rightarrow 0 \text{ a.a. in } ((0, T) \times B) \setminus Q_T. \end{cases}$$

Denoting $\{\varrho_\omega, \mathbf{u}_\omega\}$ the corresponding solution constructed in the previous section, we deduce that

$$\int_0^T \int_{\Omega_t} \left| \nabla_x \mathbf{u}_\omega + \nabla_x^t \mathbf{u}_\omega - \frac{2}{3} \operatorname{div}_x \mathbf{u}_\omega \mathbb{I} \right|^2 \, dx \, dt < c, \quad \int_0^T \int_{B \setminus \Omega_t} \mu_\omega \left| \nabla_x \mathbf{u}_\omega + \nabla_x^t \mathbf{u}_\omega - \frac{2}{3} \operatorname{div}_x \mathbf{u}_\omega \mathbb{I} \right|^2 \, dx \, dt \leq c, \quad (3.19)$$

where the latter estimates yields

$$\int_0^\tau \int_{B \setminus \Omega_t} \mu_\omega \left(\nabla_x \mathbf{u}_\omega + \nabla_x^t \mathbf{u}_\omega - \frac{2}{3} \operatorname{div}_x \mathbf{u}_\omega \mathbb{I} \right) : \nabla_x \boldsymbol{\varphi} \, dx \, dt \rightarrow 0 \text{ as } \omega \rightarrow 0$$

for any fixed $\boldsymbol{\varphi}$.

On the other hand, as we know from Lemma 3.1 that the density ϱ_ω is supported by the “fluid” region Q_T , we can still use (3.19), together with Korn’s inequality to obtain $\int_0^T \int_{\Omega_t} |\nabla_x \mathbf{u}_\omega|^2 \, dx \, dt \leq c$.

Repeating step by step the arguments of the preceding section, we let $\omega \rightarrow 0$ to obtain the momentum equation in the form

$$\begin{aligned}
&\int_{\Omega_\tau} \varrho \mathbf{u} \cdot \boldsymbol{\varphi}(\tau, \cdot) \, dx - \int_{\Omega_0} (\varrho \mathbf{u})_{0,\delta} \boldsymbol{\varphi}(0, \cdot) \, dx \\
&= \int_0^\tau \int_{\Omega_t} \left(\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho [\mathbf{u} \otimes \mathbf{u}] : \nabla_x \boldsymbol{\varphi} + p(\varrho) \operatorname{div}_x \boldsymbol{\varphi} + \delta \varrho^\beta \operatorname{div}_x \boldsymbol{\varphi} - \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \boldsymbol{\varphi} \right) \, dx \, dt
\end{aligned} \quad (3.20)$$

for any test function $\boldsymbol{\varphi}$ as in (3.12). Note that compactness of the density is now necessary only in the “fluid” part Q_T so a possible loss of regularity of $\mathbf{u}_\omega$ outside Q_T is irrelevant.

3.2.3 Vanishing artificial pressure: $\delta \rightarrow 0$

The final step is standard, we let $\delta \rightarrow 0$ in (3.20) to get rid of the artificial pressure term $\delta \varrho^\beta$ and to adjust the initial conditions, see [4, Chapter 6]. However, the momentum equation identity (3.20) holds only for the class of functions specified in (3.12). The last step of the proof of Theorem 2.1 is therefore to show that the class of admissible test functions can be extended by density arguments. For more details, [7].

Acknowledgement

This work has been supported by the Czech Science Foundation (GAČR) through project 22-01591S.

References

- [1] K. Bhandari, B. Huang, and Š. Nečasová. Weak solutions to the heat conducting compressible self-gravitating flows in time-dependent domains. *arXiv:2307.09348*, (to appear in) *Math. Models Methods Appl. Sci.*, 2024.
- [2] R. J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, 98(3):511–547, 1989.
- [3] A. Feireisl, E. and Novotný. *Singular limits in thermodynamics of viscous fluids*. Advances in Mathematical Fluid Mechanics. Birkhäuser/Springer, Cham, Second edition, 2017.
- [4] E. Feireisl. *Dynamics of viscous compressible fluids*, volume 26 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2004.
- [5] E. Feireisl. On the motion of a viscous, compressible, and heat conducting fluid. *Indiana Univ. Math. J.*, 53:1707–1740, 2004.
- [6] E. Feireisl, O. Kreml, V. Mácha, and Š. Nečasová. On the low Mach number limit of compressible flows in exterior moving domains. *J. Evol. Equ.*, 16:705–722, 2016.
- [7] E. Feireisl, O. Kreml, Š. Nečasová, J. Neustupa, and J. Stebel. Weak solutions to the barotropic Navier-Stokes system with slip boundary conditions in time dependent domains. *J. Differential Equations*, 254(1):125–140, 2013.
- [8] E. Feireisl, O. Kreml, Š. Nečasová, J. Neustupa, and J. Stebel. Incompressible limits of fluids excited by moving boundaries. *SIAM J. Math. Anal.*, 46(2):1456–1471, 2014.
- [9] E. Feireisl, J. Neustupa, and J. Stebel. Convergence of a Brinkman-type penalization for compressible fluid flows. *J. Differential Equations*, 250(1):596–606, 2011.
- [10] E. Feireisl, A. Novotný, and H. Petzeltová. On the existence of globally defined weak solutions to the Navier-Stokes equations. *J. Math. Fluid Mechanics*, 3(4):358–392, 2001.
- [11] M. Kalousek, S. Mitra, and Š. Nečasová. Existence of weak solution for a compressible multi-component fluid structure interaction problem. *arXiv preprint arXiv:2301.11216*, (to appear in) *J. Math. Pures Appl.*, 2024.
- [12] O. Kreml, V. Mácha, Š. Nečasová, and A. Wróblewska-Kamińska. Flow of heat conducting fluid in a time-dependent domain. *Z. Angew. Math. Phys.*, 69(5):Paper No. 119, 27, 2018.
- [13] O. Kreml, V. Mácha, Š. Nečasová, and A. Wróblewska-Kamińska. Weak solutions to the full Navier-Stokes-Fourier system with slip boundary conditions in time dependent domains. *J. Math. Pures Appl. (9)*, 109:67–92, 2018.
- [14] O. Kreml, Š. Nečasová, and T. Piasecki. Local existence of strong solutions and weak-strong uniqueness for the compressible Navier-Stokes system on moving domains. *Proc. Royal Soc. Edinburgh A*.
- [15] P.-L. Lions. *Mathematical Topics in Fluid Mechanics: Volume 2: Compressible Models*, volume 2. Oxford University Press, 1996.
- [16] V. Mácha, B. Muha, Š. Nečasová, A. Roy, and S. Trifunović. Existence of a weak solution to a nonlinear fluid-structure interaction problem with heat exchange. *Comm. Partial Differential Equations*, 47(8):1591–1635, 2022.
- [17] Y. Stokes and G. Carrey. On generalised penalty approaches for slip, free surface and related boundary conditions in viscous flow simulation. *Inter. J. Numer. Meth. Heat Fluid Flow*, 21:668–702, 2011.