

INTERNAL WAVE TURBULENCE IN LARGE ASPECT RATIO DOMAINS

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Abstract

Continuous stratification profoundly impacts the onset of hydrodynamical instabilities due to a unique dispersion relation. According to this relation, wave beams and wave energy propagate perpendicular to the phase velocity. It has been also observed that the dynamics of wave beams (webs of rays) in closed domains differ considerably from conventional ray tracing. In a general case, an attracting trajectory exists for the wave beams, referred to as wave attractors. These trajectories evolve due to the focusing/defocusing of wave packets upon reflection from the walls [1].

Owing to the high concentration of wave energy along these attracting paths, they are particularly susceptible to instabilities and serve as a source for the propagation of secondary waves. In our previous works, we demonstrated that if the aspect ratio of the motion is of the order of one, the transitions to turbulence can be effectively described using a laboratory toy-box model [2]. In such a laboratory model, turbulence originates from cascades of triadic resonances. This model represents one important case of natural flows. Another case involves large aspect ratio domains, where the horizontal scale is much larger than the vertical, yet buoyancy effects cannot be neglected.

Previously, we described the peculiarities of linear regimes of large aspect ratio wave attractors and their instabilities at multiples of half the forcing frequency. Now, we present a full scenario of weak (not reaching overturning) instabilities, encompassing superharmonic and subharmonic frequencies at half the forcing frequency, along with cascades of triadic resonance instabilities in each half-harmonic frequency interval.

Keywords: internal waves, inertial waves, wave attractors, wave turbulence

1 Introduction

Geophysical turbulence exhibits distinct characteristics compared to conventional turbulence flows, primarily due to the effects of stratification and/or rotations which are different at various scales of flows. We will focus on the small-scale effects where the direct action of buoyancy significantly affects momentum equations and cannot be neglected, as opposed to quasi-hydrostatic modeling at large scales.

In continuously stratified fluids the gravity waves propagate mostly by beams of waves. A very unusual property of internal wave beams in a continuously stratified medium is that they propagate perpendicular to the phase velocity. This property follows from the dispersion relation of the internal (and similar for the inertial) waves. The frequency of a wave depends only on the direction of a plane wave and does not define any length scale:

$$\frac{\omega}{N} = \sin(\theta) \quad (1)$$

where ω - angular wave frequency, N buoyancy frequency, and θ angle of the wave propagation with direction of gravity.

When two wave beams reflect from a surface, which is inclined with respect to gravity, the distance between them may increase or decrease, as can be seen in Fig. 1. Remarkably, it was found that for a closed geometry with inclined boundaries focusing dominates over defocusing and this results in emergence of limit cycles in physical space, to which all the wave beams are attracted after consecutive reflections from the boundaries.

Wave attractors can be defined as limit cycles for beams of internal or inertial waves generated by monochromatic external forcing in stratified or rotating fluids. Ray tracing methodology has demonstrated that, in a general case of a domain with inclined walls, a wave attractor exists, except for specific instances of global resonances [1]. On the other hand, if the wave attractor has many reflections from the walls, and the web of attractor rays become quite dense in the domain, the time of establishment of this attractor regime would be too long in real fluid [4], so they are hardly possible to see both in laboratory and in simulations.

In case of rotating homogeneous fluids, a very similar dispersion relation holds for inertial waves

$$\frac{\omega}{2\Omega} = \cos(\theta) \quad (2)$$

hence all the mentioned conclusions on wave beam dynamics keep validity for inertial waves. In [3] the inertial wave attractors in annular layers with trapezoidal section were modeled for the first time with DNS. Cascades of triadic instabilities were revealed and analyzed. This study was made only for a conventional case of equal scales of motion in direction of the axis of rotation and in the perpendicular plane.

In ideal fluids wave attractors represent a kind of singularity, since there is an infinitely high concentration of the wave beams approaching the attractor. In real viscous fluids, wave attractors are formed from a balance between geometrical focusing and the diffusion of momentum and density. If the web of rays is densely packed, the time required to establish the final regime may be impractically long to achieve, so the most relevant case for the study of effects of accumulation of energy along wave attractors is the (1,1) attractors with one reflection from a horizontal boundary and one from a vertical. An important case of n horizontal reflection and one vertical is considered in [4] with estimation of the time of establishment. As was shown above most laboratory and numerical studies have focused on cases where vertical and horizontal motions are of the same order.

Besides energy transfer across long distances, and transformation of global tidal motions to the small-scale turbulence, internal/inertial waves maybe responsible for vertical mixing and changing of stratification profile. In [6, 2] effects of mixing due to internal wave turbulence were estimated with the help of probability density function of vorticity and its analysis with respect to Miles-Howard criterion of stability, and also conformity of wave vectors fields to the dispersion relation.

In our current investigation of stratified turbulence in horizontally elongated domains, we will show that in viscous fluids the aspect ratio can play an important role in both energy accumulation and the process of development of instabilities and turbulence.

2 Laboratory/mathematical setup and analysis

The mathematical setup is analogous to the one described in [5] and earlier in [6] for aspect ratio close to one.

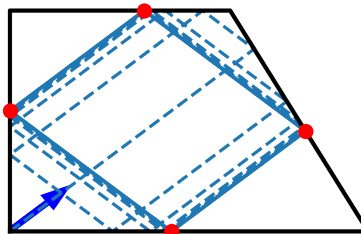


Figure 1: Formation of a wave attractor after multiple reflections of a wave beam from the walls of the trapezoid.

The stratified fluid is confined in trapezoid container with two horizontal boundaries, the left boundary is vertical and the right wall is inclined with respect to the vertical by the angle α . Previous studies of (1,1) waves attractors mostly considered the height and length to be roughly

of the same order, f.e. $H = 40$ cm and $L = 1.5H = 60$ cm. For a (1,1) attractor it means also approximately the same scales of motion in vertical in horizontal directions. Many important aspects of such a setup, as including scale effects, mixing and conformity to dispersion relation were considered in previous studies ([7, 2, 8, 9, 10]) Below we keep $H = 40$ cm, and L will be equal either to 60 cm or 300 cm. Since the dimensional analysis is a subject of debate for such a laboratory setup, we will stick to dimensional setup (in CGS units) while giving all the important parameters and results in nondimensional form.

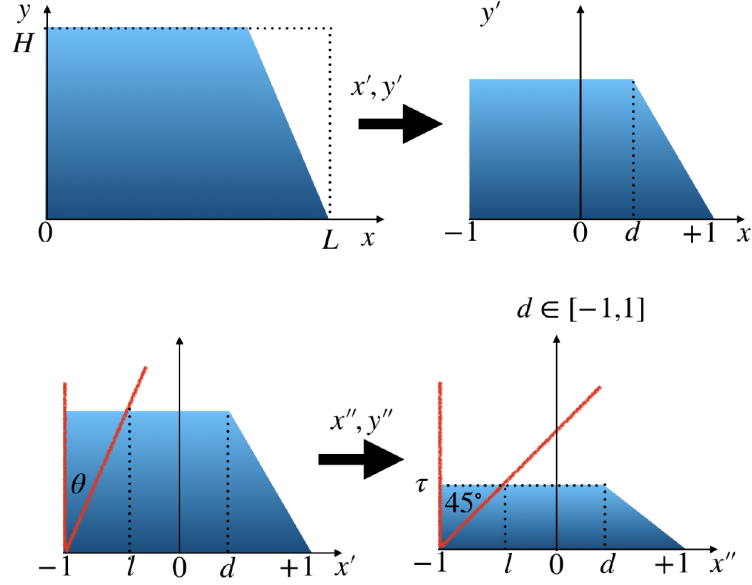


Figure 2: Sketch of the laboratory/numerical setup and transformation from original coordinates to (d, τ) plane.

The Boussinesq approximation [11] works very well in this case since the drop of the density over the height will not exceed five percent and also the vertical scale of motion is by orders of magnitude less than the total height. In Boussinesq approximation the momentum balance and mass conservation take the form:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho_m} \nabla \tilde{p} + \frac{\rho_s}{\rho_m} \vec{g} + \nu \Delta \vec{v} \quad (3)$$

$$\nabla \cdot \vec{v} = 0. \quad (4)$$

where $\rho = \rho_m + \rho_s$, ρ_m is the minimal density (density of “fresh” water), and ρ_s the partial density of the dissolved salts, subject to the following equation:

$$\frac{\partial \rho_s}{\partial t} + \vec{v} \cdot \nabla \rho_s = \nabla \cdot \frac{\nu}{Sc} \nabla \rho_s, \quad (5)$$

Sc is the Schmidt number, so ν/Sc corresponds to the salts diffusivity, $\tilde{p} = p - \rho_m g z$.

The energy is injected to the system via the periodically oscillating upper boundary. In such a way the upper boundary serves as an analogue to a global tidal impact. Since the amplitude of the oscillation is small compared to the height ($< 0.01H$) we will model the oscillations with the help of the boundary condition on the vertical velocity:

$$\vec{v}|_{y=H} = a\omega_0 \sin(\omega_0 t) \sin\left(\pi \frac{x}{L}\right) \vec{e}_y. \quad (6)$$

On all the other boundaries we have no-slip condition. The equations (3, 4, 5, 6) together with no-slip boundary conditions form a well-posed initial boundary value problem.

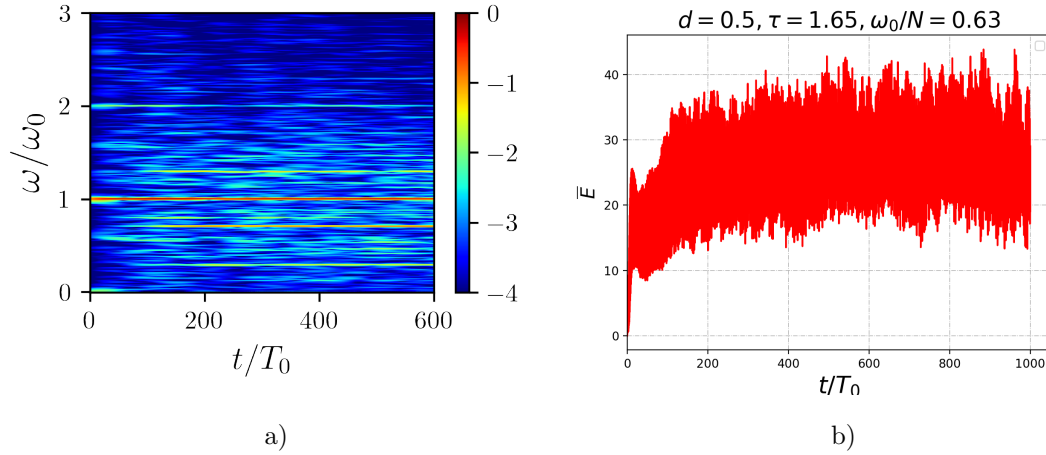


Figure 3: Time frequency diagram of the turbulent motion in a box model with aspect ratio of order one, $\varepsilon = 0.66$, and plot of averaged relative kinetic energy, defined as in [12, 5]

In case of an ideal fluid the mathematical setup described above involves five dimensional parameters (ω, N, α, H, L). With the help of a horizontal translation followed by horizontal/vertical compression they can be transformed to a set of just two parameters: (d, τ) [1]. d and τ are nondimensional, the first corresponds to the location of the slope, and second is the nondimensional height after vertical compression in such a way that the wavebeams have an angle 45° with the gravity, so the parameter τ is actually the result of nondimensionalisation of the forcing frequency. Transformation from the original laboratory dimensions to (d, τ) plane is represented in Fig. 2. For an ideal fluid dynamics of internal/inertial wave beams are completely defined by these two parameters. In particular, all the four geometries show in Fig. 2 will give absolutely same result. This will not be the case if these four geometries are filled with a viscous fluid.

For a viscous and diffusive flow there appear additional dimensionless parameters. The first is $\widetilde{Re} = \frac{H a \omega_0}{\nu}$. Though it may be interpreted as an analogy to the Reynolds number, it does not play exactly the same role as in the shear flows like pipe or Couette flows. Due to the ability of the attractor to accumulate the kinetic energy, the shear velocity can grow by orders of magnitude as compared to the velocity at the wavemakers, hence the difficulty of the use of $\widetilde{Re} = \frac{H a \omega_0}{\nu}$ as a measure of the shear motion in this study. The second parameter is the aspect ratio $\frac{L}{H}$, it is inverse to $\varepsilon = \frac{H}{L}$, which is often referred to as the “hydrostatic” nondimensional parameter, if H and L are also the scales of the horizontal and vertical velocities [13].

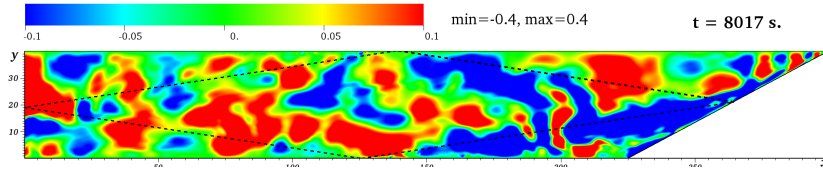


Figure 4: Instantaneous snapshot of vertical velocity for $\varepsilon = 0.13$

Nonlinear effects are important for the described problem even in laminar regimes due to the nonlinear nature of the interaction of the wave beams with boundaries. Hence the best numerical approach would be a kind of spectral decomposition over some full system of functions, satisfying boundary conditions, as f.e. pseudospectral approach with Fourier/Chebyshev functions. Here application of such an approach is practically impossible due to difficulties in finding eigenfunctions for non-orthogonal geometries. Most high-order finite-difference schemes also rely on orthogonal geometry. This is why in this case a compromise can be achieved with a combination of finite-elements ap-

proach and spectral decomposition over full system in each element, which gave the name to the spectral-element approach. Spectral elements method described in [14, 15] allows to overcome difficulties with non-orthogonal geometry, and at the same time suitable for study of highly nonlinear effects. In this paper we will stick to the same approach for numerical simulation as given in [6].

In [5] we have made a comparative analysis of small amplitude wave attractor for aspect ratio close to one and large aspect ratio. The major described distinction was found in a degree of concentration of kinetic energy for final regime. Next we have shown the appearance of waves at integer superharmonics and subharmonic instability at half forcing frequency. Before describing the fully developed wave turbulent regime we give an example of fully developed regime for aspect ratio close to unity. In Fig. 3 a) time-frequency diagram shows a complex nonlinear dynamics of cascades of TRI below the forcing frequency, and Fig. 3 b) shows corresponding fully stochastic behaviour of kinetic energy with concentration of energy about 30 times higher as compared to the kinetic energy of the wave maker (definition of $\bar{E}$ can be found in [12, 5]).

The typical field of vertical velocity for large aspect ratio motion is shown in Fig. 4. One can notice that without contour of theoretic linear wave attractor, it is hardly possible to say anything on existence of an attracting trajectory. Yet time-frequency diagram and plot of kinetic energy show gigantic accumulation of wave energy [5]. Filtering of velocity field at frequency of external forcing f_0 shows the well defined structure of the (1,1) wave attractor, and in the vicinity of this structure most of the kinetic energy in the tank is concentrated. So even though the flows seems to be absolutely turbulent, really (1,1) structure continue to play a major role in forming of the final regime.

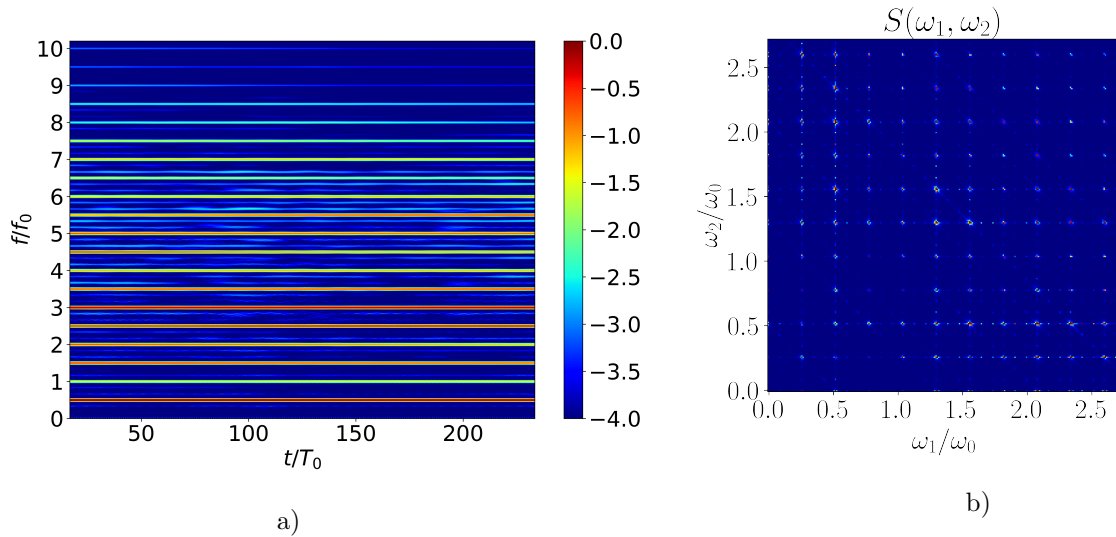


Figure 5: a) Time-frequency diagram of vertical velocity for large aspect ratio domain and amplitude, for which the regime with daughter waves at multiple integers of half-frequency described in [5] loses stability. The corresponding bispectra b) clearly exhibits TRI located along the anti diagonals in squares with $f_0/2$ side.

For large aspect ratio regime with $\epsilon = 0.13$ the development of the regime at a higher amplitude reveals enrichment of spectrum with TRI in each half- f_0 interval as shown in Fig. 5 with the help of time-frequency diagram and bispectra. Together with the results from [5] it represents the complete scenario of transition to fully developed "weak turbulence", which in this case corresponds to the turbulent motion which does not yet achieve overturning and associated mixing. An important consequence of this alternative scenario to subharmonic TRI cascades are the changes in spacial spectra and also different coherent structures resulting from the subsequent inverse cascades.

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