

SECOND ORDER SEMI-DISCRETIZATION FOR UNSTEADY STOKES FLOW

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Abstract

We consider a second order implicit time stepping procedure for the unsteady Stokes equations in bounded domains of $\mathbb{R}^3$. Using energy estimates we prove optimal convergence properties in a series of Sobolev spaces $H^m(G)$ ($m = 0, 1, 2$) uniformly in time, provided that the Stokes solution has a high degree of regularity uniformly in time. Here in the case of our second order method the solution of the Stokes equations has to satisfy a certain non-local compatibility condition at the parabolic boundary being virtually uncheckable for given data, which can be satisfied by a special initial construction.

Keywords: Unsteady Stokes equations, semi-discretization of 2nd order, convergence

1 Introduction

Initial boundary value problems for parabolic equations can be reduced to boundary value problems for equations of elliptic type, if finite differences in time are used. In the present paper we apply elementary energy estimates to prove optimal convergence properties of an implicit second order time stepping procedure for the unsteady Stokes equations

$$\begin{aligned} \partial_t v - \nu \Delta v + \nabla p &= f \\ &\text{in } (0, T) \times G, \\ \operatorname{div} v &= 0 \\ v|_{\partial G} &= 0, \quad v|_{t=0} = v_0. \end{aligned} \tag{1.1}$$

These equations are important in hydrodynamics. They describe the motion of a viscous incompressible fluid, if the nonlinear term $v \cdot \nabla v$ of the corresponding Navier-Stokes equations is ignorable small. In (1.1) the vector $v = (v_1(t, x), v_2(t, x), v_3(t, x))$ is the velocity field and the scalar $p = p(t, x)$ the kinematic pressure function of the fluid at time $t \in (0, T)$ in $x \in G$. The constant $\nu > 0$ is the kinematic viscosity, and the external force density f together with the initial velocity v_0 are given data. The condition $\operatorname{div} v = 0$ means the incompressibility of the fluid and $v|_{\partial G} = 0$ expresses that the fluid adheres to the boundary ∂G (no-slip condition). Throughout this paper we consider (1.1) in a fixed cylindrical domain $(0, T) \times G$, where $0 < T < \infty$ is given and $G \subset \mathbb{R}^3$ is a bounded domain with a sufficiently smooth compact boundary ∂G .

We consider an implicit time stepping procedure of second order for (1.1): Setting

$$h = \frac{T}{N} > 0, \quad t_k = k \cdot h \quad (k = 0, 1, \dots, N),$$

we want to approximate the solution v, p of (1.1) at time t_k by the solution v^k, p^k ($k = 1, 2, \dots, N$) of a boundary value problem arising as follows: Addition of the implicit scheme

$$\frac{(v^k - v^{k-\frac{1}{2}})}{h/2} - \nu \Delta v^k + \nabla p^k = \frac{1}{h/2} \int_{(k-\frac{1}{2})h}^{kh} f(t) dt$$

and the explicit scheme

$$\frac{(v^{k-\frac{1}{2}} - v^{k-1})}{h/2} - \nu \Delta v^{k-1} + \nabla p^{k-1} = \frac{1}{h/2} \int_{(k-1)h}^{(k-\frac{1}{2})h} f(t) dt$$

leads to a second order procedure for (1.1). Here we approximate the solution v, p at time t_k by the solution v^k, p^k ($k = 1, 2, \dots, N$) of

$$\begin{aligned} \frac{(v^k - v^{k-1})}{h} - \frac{\nu}{2} \Delta(v^k + v^{k-1}) + \frac{1}{2} \nabla(p^k + p^{k-1}) &= \frac{1}{h} \int_{(k-1)h}^{kh} f(t) dt, \\ \operatorname{div} v^k &= 0, \quad v^k|_{\partial G} = 0, \quad v^0 = v_0 \quad \text{in } G. \end{aligned} \quad (1.2)$$

This method is implicit for the sum $(v^k + v^{k-1})$, and we can prove similar convergence statements as for first order methods. Moreover, we show (see Theorem 3.3)

$$\max_k \|v^k - v(t_k)\|_{H^{2-i}(G)} = O(h^{1+\frac{i}{2}}) \quad \text{as } h \rightarrow 0, \quad i = 0, 1, 2,$$

provided that the solution v is strongly continuous from $[0, T]$ in the Sobolev space $H^{2+i}(G)$. It is known (Temam [7]), however, that such an assumption is not realistic in general, not even if the data f and v_0 are smooth: Heywood and Rannacher [3] have shown (see also Rautmann [5]) that any solution v of (1.1) being strongly $H^3(G)$ – continuous in $[0, T]$, has to satisfy a non-local compatibility condition at time $t = 0$, which is uncheckable for given data. Nevertheless, we can prove the above assertions: If, instead of the initial velocity, the initial acceleration $a_0 = \partial_t v|_{t=0}$ is prescribed such that its trace $a_0|_{\partial G}$ on the boundary ∂G vanishes, then the corresponding Stokes solution has the above required continuity property (see Proposition 2.3).

2 Notation and preliminaries

Throughout this paper, $G \subset \mathbb{R}^3$ is a bounded domain with a compact boundary ∂G of class C^4 . In the following, all (vector-) functions are real valued. As usual, $C_0^\infty(G)$ denotes the spaces of smooth functions defined in G with compact support, and $L^2(G)$ is the Lebesgue (Hilbert) space equipped with scalar product and norm

$$\langle f, g \rangle = \int_G f(x)g(x)dx, \quad \|f\| = \langle f, f \rangle^{\frac{1}{2}},$$

respectively. The Sobolev (Hilbert) space $H^m(G)$, $m \in \mathbb{N} = \{0, 1, 2, \dots\}$, is the space of functions f such that $\partial^\alpha f \in L^2(G)$ for all $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$ with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \leq m$. Its norm is denoted by

$$\|f\|_m = \|f\|_{H^m(G)} = \left(\sum_{|\alpha| \leq m} \|\partial^\alpha f\|^2 \right)^{\frac{1}{2}}, \quad \partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3},$$

where $\partial_k = \frac{\partial}{\partial x_k}$ ($k = 1, 2, 3$) is a distributional derivative. The completion of $C_0^\infty(G)$ with respect to $\|\cdot\|_m$ is denoted by $H_0^m(G)$ ($H_0^0(G) = H^0(G) = L^2(G)$). The spaces $C_0^\infty(G)^3$, $L^2(G)^3$, $H^m(G)^3, \dots$ are the corresponding spaces of vector fields $u = (u_1, u_2, u_3)$. Here norm and scalar product are denoted as in the scalar case, hence, for example,

$$\langle u, v \rangle = \sum_{k=1}^3 \langle u_k, v_k \rangle, \quad \|u\| = \langle u, u \rangle^{\frac{1}{2}} = \left(\int_G |u(x)|^2 dx \right)^{\frac{1}{2}},$$

where $|u(x)| = (u_1(x)^2 + u_2(x)^2 + u_3(x)^2)^{\frac{1}{2}}$ is the Euclidian norm of $u(x) \in \mathbb{R}^3$. The completion of

$$C_{0,\sigma}^\infty(G)^3 = \{u \in C_0^\infty(G)^3 \mid \operatorname{div} u = 0\}$$

with respect to the norm $\|\cdot\|$ and $\|\cdot\|_1$ are basic spaces for the treatment of the Stokes equations and denoted by $H(G)$ and $V(G)$, respectively. In $H_0^1(G)^3$ and $V(G)$ we also use

$$\langle \nabla u, \nabla v \rangle = \sum_{k,j=1}^3 \langle \partial_k u_j, \partial_k v_j \rangle, \quad \|\nabla u\| = \langle \nabla u, \nabla u \rangle^{\frac{1}{2}}$$

as scalar product and norm. Moreover, we need the B -valued spaces $C^m(J, B)$ and $H^m(a, b, B)$, $m \in \mathbb{N}$, where $J \subset \mathbb{R}$ is a compact interval, where $a, b \in \mathbb{R}$ ($a < b$), and where B is any of the spaces above. In case of $C^0(\cdot, \cdot)$ we simply write $C(\cdot, \cdot)$, and we sometimes use $H, V, H^m, \dots$ instead of $H(G), V(G), H^m(G), \dots$, if the domain of definition is clear from the context. Let

$$P : L^2(G)^3 \longrightarrow H(G)$$

denote Weyl's orthogonal projection such that

$$L^2(G)^3 = H(G) \oplus \{v \in L^2(G)^3 \mid v = \nabla p \text{ for some } p \in H^1(G)\}.$$

Because P commutes with the strong time derivative ∂_t , from the Stokes equations (1.1) we obtain the following evolution equations for the function $(0, T) \ni t \rightarrow v(t) \in H(G)$:

$$\partial_t v(t) - \nu P \Delta v(t) = Pf(t), \quad v(0) = v_0. \quad (2.1)$$

In this case, the condition $\operatorname{div} v = 0$ and the boundary condition $v|_{\partial G} = 0$ are satisfied in the sense that we require $v(t) \in V(G)$ for $t \in (0, T)$. In the following proposition we collect some known facts ([2, 7, 8]) about the solvability of the projected equations (2.1).

2.1 Proposition: *Let $v_0 \in H^2(G)^3 \cap V(G)$ and $f \in H^1(0, T, H(G))$. Then there is a unique solution v of (2.1) in $(0, T) \times G$ such that $v \in C([0, T], H^2(G)^3 \cap V(G))$ and $\partial_t v \in C([0, T], H(G)) \cap L^2(0, T, H^1(G)^3)$. Moreover, there is some constant $K_1 = K_1(G, \nu, f, v_0)$ independent of $t \in [0, T]$ with*

$$\int_0^t \|\nabla \partial_\tau v(\tau)\|^2 d\tau \leq K_1, \quad \|v(t)\|_2 \leq K_1, \quad \|\partial_t v(t)\| \leq K_1 \quad (t \in [0, T]).$$

The property $v \in C([0, T], H^2(G)^3 \cap V(G))$ is the highest possible spatial regularity uniformly in time for any solution v of (2.1), if integer order Sobolev (Hilbert) spaces are used [5]. The next proposition (see [3, 7]) shows that higher order spatial regularity uniformly in time is possible only, if an additional compatibility condition is satisfied.

2.2 Proposition: *Let v_0 and f be given as in Proposition 2.1, and let v denote the solution of the Stokes equations (2.1) from Proposition 2.1. If in addition $v_0 \in H^4(G)^3 \cap V(G)$ and $f \in H^1(0, T, H^2(G)^3 \cap H(G))$ with $\partial_t^2 f \in L^2(0, T, H(G))$, then even $v \in C([0, T], H^4(G)^3 \cap V(G))$ with $\partial_t v \in C([0, T], H^2(G)^3 \cap V(G))$ and $\partial_t^2 v \in C([0, T], H(G)) \cap L^2(0, T, H^1(G)^3)$ if and only if*

$$(\nu P \Delta v_0 + f(0)) \Big|_{\partial G} = 0. \quad (2.2)$$

In this case there is a constant $K_2 = K_2(G, \nu, f, v_0)$ independent of $t \in [0, T]$ with

$$\begin{aligned} \int_0^t \|\nabla \partial_\tau^2 v(\tau)\|^2 d\tau &\leq K_2, & \|v(t)\|_4 &\leq K_2, \\ \|\partial_t v(t)\|_2 &\leq K_2, & \|\partial_t^2 v(t)\| &\leq K_2 \quad (t \in [0, T]). \end{aligned}$$

Because $f = Pf$, from (2.1) we see that the condition (2.2) means exactly

$$\partial_t v(0) \Big|_{\partial G} = 0. \quad (2.3)$$

This corresponds to the condition $v(0)|_{\partial G} = 0$, if we differentiate the Stokes equations (2.1) one time with respect to t and consider the resulting equations as an initial value problem for the acceleration $\partial_t v$. Thus (2.3) is satisfied, if we prescribe an initial acceleration $a_0 \in V$.

2.3 Proposition: Let $f \in H^1(0, T, H^2(G)^3 \cap H(G))$ with $\partial_t^2 f \in L^2(0, T, H(G))$ as in Proposition 2.2, and let $a_0 \in H^2(G)^3 \cap V(G)$. Then there is a unique solution v_0 of the stationary Stokes equations

$$-\nu P \Delta v_0 = (f(0) - a_0) \quad \text{in } G \quad (2.4)$$

such that $v_0 \in H^4(G)^3 \cap V(G)$. The corresponding solution v of the unsteady equations (2.1) with v_0 as initial value satisfies the compatibility condition (2.2), hence it has all regularity properties asserted in Proposition 2.2 and satisfies all estimates given there.

3 Second order discretization

In contrast to a first order method, the second order procedure considered here requires more regularity of the data f, v_0 in order to obtain second order convergence. Let us first assume $v_0 \in H^2(G)^3 \cap V(G)$ and $f \in L^2(0, T, H(G))$. Then, using the projector P , the system (1.2) reduces to ($h = T/N > 0$)

$$(v^k - v^{k-1}) - h \frac{\nu}{2} P \Delta (v^k + v^{k-1}) = \int_{(k-1)h}^{kh} f(t) dt, \quad v^0 = v_0. \quad (3.1)$$

Using well-known results from operator theory [2] and regularity statements for elliptic systems [1] the following statements can be proved.

3.1 Proposition: (a) Let $v_0 \in H^2(G)^3 \cap V(G)$ and let $f \in L^2(0, T, H(G))$. Then there is a unique solution $v^k \in H^2(G)^3 \cap V(G)$ of the system (3.1) for all $k = 1, 2, \dots, N$.

(b) If, in addition, $f \in L^2(0, T, H^2(G)^3 \cap H(G))$, then $(v^k + v^{k-1}) \in H^4(G)^3 \cap V(G)$ for all $k = 1, 2, \dots, N$.

(c) If even $v_0 \in H^4(G)^3 \cap V(G)$ and $f \in L^2(0, T, H^2(G)^3 \cap H(G))$, then $v^k \in H^4(G)^3 \cap V(G)$ for all $k = 1, 2, \dots, N$.

To prove convergence statements for our method we investigate the discretization error

$$w^k = v^k - v(t_k) \quad (k = 0, 1, 2, \dots, N).$$

Here v^k is the solution of the system (3.1) and $v(t_k)$ is the solution of the Stokes equations (2.1) on $[0, T]$ at time $t_k = kh$, $h = T/N > 0$. Using (2.1) and (3.1), a short calculation yields the identity

$$\begin{aligned} & \frac{w^k - w^{k-1}}{h} - \frac{\nu}{2} P \Delta (w^k + w^{k-1}) \\ &= \frac{\nu}{h} \int_{(k-1)h}^{kh} \left(\frac{t_k - t}{h} P \Delta v(t_{k-1}) + \frac{t - t_{k-1}}{h} P \Delta v(t_k) - P \Delta v(t) \right) dt. \end{aligned}$$

Thus we have

$$(w^k - w^{k-1}) - \frac{h\nu}{2} P \Delta (w^k + w^{k-1}) = -h\nu P \Delta G^k, \quad (3.2)$$

where G^k is defined by

$$G^k = h^{-2} \int_{(k-1)h}^{kh} \left((t_k - t)(v(t) - v(t_{k-1})) - (t - t_{k-1})(v(t_k) - v(t)) \right) dt. \quad (3.3)$$

Now using

$$v(t) = v(t_{k-1}) + \int_{(k-1)h}^t \partial_\tau v(\tau) d\tau, \quad v(t_k) = v(t) + \int_t^{kh} \partial_\tau v(\tau) d\tau,$$

from (3.3) we obtain

$$G^k = h^{-2} \int_{(k-1)h}^{kh} \left((t_k - t) \int_{(k-1)h}^t \partial_\tau v(\tau) d\tau - (t - t_{k-1}) \int_t^{kh} \partial_\tau v(\tau) d\tau \right) dt. \quad (3.4)$$

Moreover, replacing in (3.3) the Taylor expansions

$$\begin{aligned} v(t) &= v(t_{k-1}) + (t - t_{k-1}) \partial_t v(t) - \int_{(k-1)h}^t (\tau - t_{k-1}) \partial_\tau^2 v(\tau) d\tau, \\ v(t_k) &= v(t) + (t_k - t) \partial_t v(t) + \int_t^{kh} (t_k - \tau) \partial_\tau^2 v(\tau) d\tau, \end{aligned}$$

we find

$$G^k = -h^{-2} \int_{(k-1)h}^{kh} \left((t_k - t) \int_{(k-1)h}^t (\tau - t_{k-1}) \partial_\tau^2 v(\tau) d\tau + (t - t_{k-1}) \int_t^{kh} (t_k - \tau) \partial_\tau^2 v(\tau) d\tau \right) dt. \quad (3.5)$$

Depending on the regularity of the solution v , the representation (3.3), (3.4) and (3.5) of G^k will be used for the following estimates of the discretization error.

3.2 Theorem: Let $v_0 \in H^2(G)^3 \cap V(G)$ and $f \in H^1(0, T, H(G))$ be given. Let v denote the solution of the unsteady Stokes equation (2.1) in $[0, T]$ from Proposition 2.1. Let $h = T/N > 0$ ($N \in \mathbb{N}$) and let v^k for $k = 1, 2, \dots, N$ ($v^0 = v_0$) be the solution of the system (3.1) as constructed in Proposition 3.1 (a). Setting $t_k = kh$, let $w^k = v^k - v(t_k)$ ($k = 0, 1, \dots, N$) denote the discretization error. Then

$$\max \|w^k\| = \mathcal{O}(h), \quad \max \|(w^k + w^{k-1})\|_1 = \mathcal{O}(h^{\frac{1}{2}}), \quad \max \|(w^k + w^{k-1})\|_2 = o(1)$$

as $h \rightarrow 0$ (or $N \rightarrow \infty$). In particular, we have

$$\|w^k\|^2 + \frac{h\nu}{4} \sum_{j=1}^k \|\nabla(w^j + w^{j-1})\|^2 \leq 4\nu K_1 h^2 \quad (3.6)$$

with the constant K_1 from Proposition 2.1, and

$$\|Z^k\|^2 + \frac{h\nu}{4} \sum_{j=2}^k \|\nabla(Z^j + Z^{j-1})\|^2 = o(1) \quad (3.7)$$

as $h \rightarrow 0$ (or $N \rightarrow \infty$), where $Z^k = \frac{w^k - w^{k-1}}{h}$ for $k = 1, 2, \dots, N$.

In the next theorem we investigate the convergence properties of the second order scheme for Stokes solutions satisfying the compatibility condition (2.2):

3.3 Theorem: Let $v \in C([0, T], H^4(G)^3 \cap V(G))$ denote the Stokes solution constructed in Proposition 2.3. Let $h = T/N > 0$ ($N \in \mathbb{N}$), and let v^k for $k = 1, 2, \dots, N$ ($v^0 = v_0$) be the solution of the second order scheme (3.1), constructed in Proposition 3.1 (c). Setting $t_k = kh$, let $w^k = v^k - v(t_k)$ ($k = 0, 1, \dots, N$) denote the discretization error. Then

$$\max \|w^k\| = \mathcal{O}(h^2), \quad \max \|w^k\|_1 = \mathcal{O}(h^{\frac{3}{2}}), \quad \max \|w^k\|_2 = \mathcal{O}(h) \quad \text{as } h \rightarrow 0.$$

More precise, we have

$$\|w^k\|^2 + \frac{h\nu}{4} \sum_{j=1}^k \|\nabla(w^j + w^{j-1})\|^2 \leq 4\nu K_2 h^4, \quad (3.8)$$

$$\|\nabla w^k\|^2 + \frac{h\nu}{4} \sum_{j=1}^k \|P\Delta(w^j + w^{j-1})\|^2 \leq 16K_2 h^3, \quad (3.9)$$

$$\nu \|P\Delta w^k\|^2 + h \sum_{j=1}^k \|\nabla Z^j\|^2 \leq 8K_2 h^2, \quad (3.10)$$

where $Z^k = \frac{w^k - w^{k-1}}{h}$ for $k = 1, 2, \dots, N$ and K_2 is the constant from Proposition 2.2.

Proof: (a) Multiplying (3.2) scalar in L^2 by the term $(w^k + w^{k-1})$ yields

$$\|w^k\|^2 - \|w^{k-1}\|^2 + \frac{h\nu}{4} \|\nabla(w^k + w^{k-1})\|^2 \leq h\nu \|\nabla G^k\|^2. \quad (3.11)$$

This time, the norm on the right hand side can be estimated using the representation (3.5) of the right hand side G^k . This leads to the estimate

$$\begin{aligned} & h\nu \|\nabla G^k\|^2 \\ & \leq \nu h^{-2} \int_{(k-1)h}^{kh} \left\| (t_k - t) \int_{(k-1)h}^t (\tau - t_{k-1}) \partial_\tau^2 \nabla v(\tau) d\tau + (t - t_{k-1}) \int_t^{kh} (t_k - \tau) \partial_\tau^2 \nabla v(\tau) d\tau \right\|^2 dt \\ & \leq 2\nu h^{-2} \int_{(k-1)h}^{kh} \left(\left\| (t_k - t) \int_{(k-1)h}^t (\tau - t_{k-1}) \partial_\tau^2 \nabla v(\tau) d\tau \right\|^2 + \left\| (t - t_{k-1}) \int_t^{kh} (t_k - \tau) \partial_\tau^2 \nabla v(\tau) d\tau \right\|^2 \right) dt. \end{aligned}$$

Because

$$\begin{aligned} \left\| (t_k - t) \int_{(k-1)h}^t (\tau - t_{k-1}) \partial_\tau^2 \nabla v(\tau) d\tau \right\|^2 & \leq h^2 \left\| \int_{(k-1)h}^{kh} (\tau - t_{k-1}) \partial_\tau^2 \nabla v(\tau) d\tau \right\|^2 \\ & \leq h^3 \int_{(k-1)h}^{kh} \|(\tau - t_{k-1}) \partial_\tau^2 \nabla v(\tau)\|^2 d\tau \\ & \leq h^5 \int_{(k-1)h}^{kh} \|\partial_\tau^2 \nabla v(\tau)\|^2 d\tau, \end{aligned}$$

and, analogously,

$$\left\| (t - t_{k-1}) \int_t^{kh} (t_k - \tau) \partial_\tau^2 \nabla v(\tau) d\tau \right\|^2 \leq h^5 \int_{(k-1)h}^{kh} \|\partial_\tau^2 \nabla v(\tau)\|^2 d\tau,$$

we obtain the desired estimate:

$$h\nu \|\nabla G^k\|^2 \leq h^4 4\nu \int_{(k-1)h}^{kh} \|\partial_\tau^2 \nabla v(\tau)\|^2 d\sigma. \quad (3.12)$$

Hence summation of (3.11) gives

$$\|w^k\|^2 + \frac{h\nu}{4} \sum_{j=1}^k \|\nabla(w^j + w^{j-1})\|^2 \leq h^4 4\nu \int_0^T \|\partial_t^2 \nabla v(t)\|^2 dt,$$

which implies (3.8) by Proposition 2.2. In particular, we find $\max \|w^k\| = 0(h^2)$ and $\max \|\nabla(w^k + w^{k-1})\|^2 = 0(h^{3/2})$ as $h \rightarrow 0$ (or $N \rightarrow \infty$).

(b) Multiplication of (3.2) by $-P\Delta(w^k + w^{k-1})$ scalar in L^2 yields

$$\begin{aligned} & \|\nabla w^k\|^2 - \|\nabla w^{k-1}\|^2 + \frac{h\nu}{2} \|P\Delta(w^k + w^{k-1})\|^2 \\ &= h\nu \langle P\Delta G^k, P\Delta(w^k + w^{k-1}) \rangle \\ &\leq 2 \left(\frac{h\nu}{4}\right)^{\frac{1}{2}} \|P\Delta(w^k + w^{k-1})\| (h\nu)^{\frac{1}{2}} \|P\Delta G^k\| \\ &\leq \frac{h\nu}{4} \|P\Delta(w^k + w^{k-1})\|^2 + h\nu \|P\Delta G^k\|^2, \end{aligned}$$

hence

$$\|\nabla w^k\|^2 - \|\nabla w^{k-1}\|^2 + \frac{h\nu}{4} \|P\Delta(w^k + w^{k-1})\|^2 \leq h\nu \|P\Delta G^k\|^2. \quad (3.13)$$

Now we have to use both representations (3.4) and (3.5) of G^k to estimate the right hand side of (3.13). Using (3.4) and differentiating the Stokes equations with respect to t we find

$$\nu P\Delta G^k = h^{-2} \int_{(k-1)h}^{kh} \left((t_k - t) \int_{(k-1)h}^t (\partial_\tau^2 v(\tau) - \partial_\tau f(\tau)) d\tau - (t - t_{k-1}) \int_t^{kh} (\partial_\tau^2 v(\tau) - \partial_\tau f(\tau)) d\tau \right) dt, \quad (3.14)$$

hence

$$\begin{aligned} \|\nu \nabla P\Delta G^k\|^2 &\leq 2h \int_{(k-1)h}^{kh} \|\nabla(\partial_\tau^2 v(\tau) - \partial_\tau f(\tau))\|^2 d\tau \\ &\leq 4h \left(\int_{(k-1)h}^{kh} \|\partial_\tau^2 \nabla v(\tau)\|^2 d\tau + \int_{(k-1)h}^{kh} \|\partial_\tau \nabla f(\tau)\|^2 d\tau \right) \\ &=: 4h (S_1 + S_2). \end{aligned} \quad (3.15)$$

On the other hand, (3.12) implies $h^{\frac{1}{2}} \|\nabla G^k\| \leq 2h^2(S_1)^{\frac{1}{2}}$, and it follows

$$\begin{aligned} h\nu \|P\Delta G^k\|^2 &= h \langle P\Delta G^k, \nu P\Delta G^k \rangle \\ &= -h \langle \nabla G^k, \nu \nabla P\Delta G^k \rangle \\ &\leq h \|\nabla G^k\| \|\nu \nabla P\Delta G^k\| \\ &\leq 4h^3(S_1)^{\frac{1}{2}}(S_1 + S_2)^{\frac{1}{2}} \\ &\leq 4h^3(S_1 + S_2), \end{aligned}$$

hence

$$h\nu \|P\Delta G^k\|^2 \leq 4h^3 \left(\int_{(k-1)h}^{kh} \|\partial_\tau^2 \nabla v(\tau)\|^2 d\tau + \int_{(k-1)h}^{kh} \|\partial_\tau \nabla f(\tau)\|^2 d\tau \right). \quad (3.16)$$

Now summing up (3.13) with (3.16), we obtain

$$\|\nabla w^k\|^2 + \frac{h\nu}{4} \sum_{j=1}^k \|P\Delta(w^j + w^{j-1})\|^2 \leq 4h^3 \int_0^T (\|\partial_t^2 \nabla v(t)\|^2 + \|\partial_t \nabla f(t)\|^2) dt,$$

which implies the desired estimate (3.9) if we choose the constant K_2 in Proposition 2.2 such that

$$\int_0^T \|\partial_t \nabla f(t)\|^2 dt \leq K_2.$$

In particular, we obtain the optimal order of convergence in H^1 , i.e. $\max \|w^k\|_1 = O(h^{\frac{3}{2}})$ as $h \rightarrow 0$ or $N \rightarrow \infty$. This follows using the inequality of Poincarè again.

(c) Finally, multiplication of (3.2) scalar in L^2 by $-P\Delta(w^k - w^{k-1})$ leads to

$$\begin{aligned} \|\nabla(w^k - w^{k-1})\|^2 + \frac{h\nu}{2} (\|P\Delta w^k\|^2 - P\Delta\|w^{k-1}\|^2) &= h\nu \langle P\Delta G^k, P\Delta(w^k - w^{k-1}) \rangle \\ &= -h\nu \langle \nabla P\Delta G^k, \nabla(w^k - w^{k-1}) \rangle \\ &\leq h\nu \|\nabla P\Delta G^k\| \|\nabla(w^k - w^{k-1})\| \\ &\leq \frac{1}{2} (h^2 \|\nu \nabla P\Delta G^k\|^2 + \|\nabla(w^k - w^{k-1})\|^2), \end{aligned}$$

hence by (3.15)

$$\begin{aligned} \|\nabla(w^k - w^{k-1})\|^2 + h\nu (\|P\Delta w^k\|^2 - P\Delta\|w^{k-1}\|^2) &\leq h^2 \|\nu \nabla P\Delta G^k\|^2 \\ &\leq 4h^3 \left(\int_{(k-1)h}^{kh} \|\partial_\tau^2 \nabla v(\tau)\|^2 d\tau + \int_{(k-1)h}^{kh} \|\partial_\tau \nabla f(\tau)\|^2 d\tau \right). \end{aligned}$$

From this we conclude

$$\|P\Delta w^k\|^2 + \frac{h}{\nu} \sum_{j=1}^k \left\| \frac{\nabla(w^j - w^{j-1})}{h} \right\|^2 \leq h^2 \frac{4}{\nu} \int_0^T (\|\partial_t^2 \nabla v(t)\|^2 + \|\partial_t \nabla f(t)\|^2) dt,$$

which implies (3.10), choosing the constant K_2 from Proposition 2.2 as before. By means of Cattabriga's estimate $\|u\|_2 \leq c_G \|P\Delta u\|$ ([1], see also [3], p. 280) this ensures a linear order of convergence in $H^2(G)$: $\max \|w^k\|_2 = O(h)$ as $h \rightarrow 0$ (or $N \rightarrow \infty$). This proves the theorem.

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