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COUPLED VIBROACOUSTIC PROBLEM INSPIRED BY HUMAN PHONATION

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Abstract

The contribution deals with the 2D vibro-acoustic model inspired by human phonation. Its aim is to demonstrate the influence of vocal fold compliance on acoustic resonant frequencies. Vibro-acoustic coupling is highlighted under specific conditions, particularly when elastic and acoustic frequencies coincide, as seen in phonation into tubes or straws of suitable lengths used in voice therapy.

Here, vocal folds vibrations are modelled using linear elasticity theory and acoustics is described by wave equation. Both coupled subproblems are formulated in the frequency domain and numerically approximated by the finite element method. In the end the numerical results of the obtained resonant frequencies and their analysis are presented.

Keywords: vibroacoustics, coupled problem, human phonation, phonation into a tube.

1 Introduction

Vibroacoustics is a multidisciplinary field combining structural mechanics and acoustics, with typical applications including the eigenfrequency analysis of water-filled tanks and the acoustic emission of vibrating engines, see [5, 7]. Depending on the strength of coupling between the acoustic medium and the usually high-density vibrating body, various cases arise. For weak coupling, such as noise emitted by a vibrating machine, the acoustic-structure interaction can often be neglected, leading to a one-way structural-acoustic coupling. However, for moderate to strong coupling, effects range from minor perturbations to significant changes in system behavior, as observed in the structural stability analysis of liquid rocket engines [2], and both acoustic and structural coupling must be considered in these scenarios. The strength of coupling depends mainly on the frequency and spatial alignment of acoustic and structural modes, as well as factors such as the density ratio of the structure and acoustic medium and the size of the interface area, [3].

This study focuses on the acoustics of the human vocal tract, where vibroacoustic coupling becomes particularly relevant in specific cases, such as voice rehabilitation exercises involving phonation into tubes or tubes immersed in water. Research [9, 8] has demonstrated that this coupling is significant when the vocal tract extension through a tube shifts its first acoustic resonance frequency closer to the first eigenfrequency of the vocal folds (VFs). Under these conditions, vibroacoustic interaction intensifies, and measurements and computations, e.g., [6], show a notable shift in the coupled system's first acoustic resonance frequency. This motivates our mathematical analysis of such interactions.

In this paper the vibrations of the vocal folds are modeled as a linear isotropic elastic body, while the wave equation is used to model the acoustic wave propagation within the vocal tract model. Standard coupling conditions are applied at the mutual interface, as described in [7]. Both subproblems are numerically discretized using the finite element method (FEM). The resulting coupled system of equations after transformation to the frequency domain represents a generalized eigenvalue problem. Nevertheless, to identify dominant acoustic modes, we adopt a transfer function approach, solving the coupled system numerically at different prescribed frequencies, as described in [7].

The first section of this paper contains a mathematical description of the vibroacoustic problem. The second section presents the numerical method as well as the transformation of the problem into the frequency domain. Numerical results of the third section include two cases – a simply coupled vibrating mass with an acoustic space and a two-dimensional model of the vocal tract extended by a tube. The final conclusion closes the paper.

2 Vibroacoustic model

Let us consider a two-dimensional coupled vibro-acoustic problem in domain Ω composed of elastic structure domain Ω^s (vocal folds) and acoustic domain Ω^a. The first part of acoustic domain represents human vocal tract of length L_1 and the following second one is a straight tube of length L_2 and diameter d_2, see Figure 1. Three mutually disjoint boundaries $\Gamma^a_{\rm Dir}$, $\Gamma^s_{\rm Dir}$ and $\Gamma^a_{\rm Neu}$ constitute the boundary of domain Ω. The common interface between domains Ω^s and Ω^a is denoted as $\Gamma_{\rm W}$, see Figure 1.

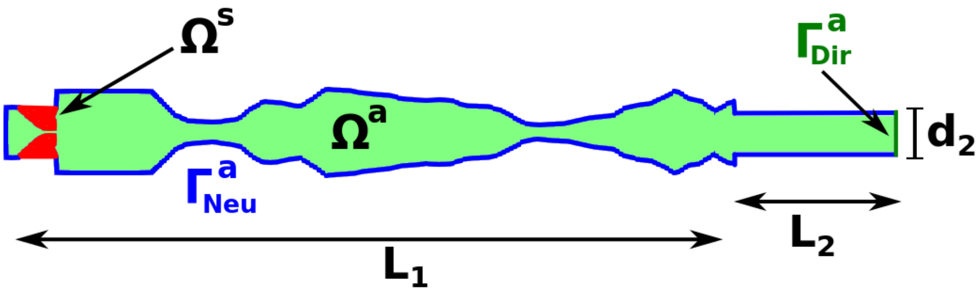

Figure 1: Scheme of vibro-acoustic domain Ω consisting of structure domain Ω^s and acoustic domain Ω^a with marked boundaries of $\partial\Omega^a$.

2.1 Elastic structure

The elastic structure vibration is described with the aid of displacement $\mathbf{u}(x,\omega) = (u_1, u_2)$ satisfying following partial differential equations

$$-\rho^s \frac{\partial^2 u_i}{\partial t^2} + \frac{\partial \tau^s_{ij}}{\partial x_j} = 0, \qquad \text{in } \Omega^{\rm s}, \quad ({\rm i} = 1, 2), \tag{1}$$

where ρ^s is the structure density and τ^s_{ij} denote the components of the Cauchy stress tensor. In the following we assume for simplicity isotropic body and then the stress tensor components can be expressed with help of the Hooke's law as

$$\tau^s_{ij} = \lambda^s \text{div}\, \mathbf{u}\, \delta_{ij} + 2\mu^s e^s_{ij}(\mathbf{u}), \tag{2}$$

where $\mathbb{I} = (\delta_{ij})$ denotes Kronecker's delta and $e^s_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$ is the small strain tensor parametrized by the Lamé coefficients λ^s, μ^s, see e.g. [1].

The elastic body is fixed at the boundary $\Gamma^s_{\rm Dir}$ and the interface condition is specified later

$$\mathbf{u}(x,t) = \mathbf{0}, \qquad \text{for } x \in \Gamma^s_{\rm Dir},\ t \in (0, {\rm T}). \tag{3}$$

2.2 Acoustics

The sound propagation through homogeneous air neglecting possible effects of background flow is described the classical wave equation

$$\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \Delta p = 0, \qquad \text{in } \Omega^a, \tag{4}$$

for unknown acoustic pressure p with notation of the speed of sound by c_0. The following boundary conditions are considered for $t \in (0, \mathrm{T})$

$$\text{a)} \;\; p(x,t) = 0, \quad \text{for } x \in \Gamma^a_{\mathrm{Dir}}, \qquad \text{b)} \;\; \frac{\partial p}{\partial \mathbf{n}^s}(x,t) = 0, \quad \text{for } x \in \Gamma^a_{\mathrm{Neu}}, \tag{5}$$

where vector $\mathbf{n}^s = (n^s_j)$ is unit inner normal to $\partial\Omega^a$. The first condition (5 a) is so called sound soft boundary condition which approximately models the free end of tube, the second condition (5 b) is the sound hard condition and it represents fully reflecting walls, see [7].

2.3 Vibroacoustic coupling

The vibroacoustic problem is coupled through boundary conditions on the interface Γ_{W}. The boundary condition for the acoustic pressure p is derived from the requirement of velocity continuity in the normal direction, see e.g. [7],

$$\frac{\partial p}{\partial \mathbf{n}^s}(x,t) = -\rho^a \frac{\partial^2 \mathbf{u}(x,t)}{\partial t^2} \cdot \mathbf{n}^s, \qquad x \in \Gamma_{\mathrm{W}}. \tag{6}$$

where ρ^a is the density of air. This boundary condition specifies the acoustic emission generated by the normal acceleration of the vibrating surface Γ_{W}.

For the elastic body, the boundary condition emerges from the stress continuity in the normal direction

$$\tau^s_{ij}(x,t)\, n^s_j = p(x,t)\, n^s_i, \qquad x \in \Gamma_{\mathrm{W}}, \tag{7}$$

where $\mathbf{n}^s = (n^s_j)$ is the unit normal to Γ_{W} pointing from Ω^s to Ω^a, consistent with the previous notation.

3 Numerical model

The elastic and acoustic problems are first weakly reformulated. They are transformed into the frequency domain and the fully coupled formulation is later discretized in space by the FEM.

3.1 Weak formulation

The elastic problem (1) is formulated weakly in standard way, see [10] and [11], together with considered boundary conditions (3) and (7) and it reads: find $\mathbf{u} \in \mathbf{V}$ such that

$$\rho^s \cdot m^s\left(\frac{\partial^2 \mathbf{u}}{\partial t^2}, \boldsymbol{\psi}\right) + k^s(\mathbf{u}, \boldsymbol{\psi}) = -c^a(p, \boldsymbol{\psi}) \tag{8}$$

is satisfied for all $\boldsymbol{\psi} \in \mathbf{V} = \{\mathbf{f} \in \mathbf{H}^1(\Omega^s) | \mathbf{f} = \mathbf{0} \text{ on } \Gamma^s_{\mathrm{Dir}}\}$. The bilinear forms $m^s(\cdot,\cdot), k^s(\cdot,\cdot)$ and $c^a(\cdot,\cdot)$ are defined by

$$m^s(\mathbf{u}, \boldsymbol{\psi}) = (\mathbf{u}, \boldsymbol{\psi})_{\Omega^s}, \quad k^s(\mathbf{u}, \boldsymbol{\psi}) = (\lambda^s (\mathrm{div}\, \mathbf{u})\, \mathbb{I} + 2\mu^s \mathbf{e}^s(\mathbf{u}), \mathbf{e}^s(\boldsymbol{\psi}))_{\Omega^s}, \quad c^a(p, \boldsymbol{\psi}) = (p\mathbf{n}^s, \boldsymbol{\psi})_{\Gamma_{\mathrm{W}}}, \tag{9}$$

and the scalar product of functions from $L^2(\mathcal{M})$ is denoted by $(\cdot,\cdot)_{\mathcal{M}}$.

Similarly, the weak formulation of acoustic problem (4) together with conditions (5) and (6) in functional space $Y = \{f \in H^1(\Omega^a) | f = 0 \text{ on } \Gamma^{\mathrm{a}}_{\mathrm{Dir}}\}$ reads: find $p \in Y$ that

$$\frac{1}{c_0^2} m^a\left(\frac{\partial^2 p}{\partial t^2}, \eta\right) + k^a(p, \eta) = \rho^a \cdot c^s\left(\frac{\partial^2 \mathbf{u}}{\partial t^2}, \eta\right) \tag{10}$$

is satisfied for any $\eta \in Y$ with bilinear forms $m^a(\cdot,\cdot), k^a(\cdot,\cdot)$ and $c^s(\cdot,\cdot)$ defined as

$$m^a(p, \eta) = (p, \eta)_{\Omega^a}, \quad k^a(p, \eta) = (\nabla p, \nabla \eta)_{\Omega^a}, \quad c^s(\mathbf{u}, \eta) = (\mathbf{u} \cdot \mathbf{n}^s, \eta)_{\Gamma_{\mathrm{W}}}. \tag{11}$$

3.2 Transformation to the frequency domain

The Fourier transform can be utilized to transformation the vibroacoustic problem into the frequency domain for general cases. Nevertheless let us assume a harmonic behaviour characterized by a single angular frequency ω. In this simplified scenario, the transformation is expressed as

$$\hat{\mathbf{u}}(x,\omega) = \mathbf{u}(x,t)e^{i\omega t}, \qquad \hat{p}(x,\omega) = p(x,t)e^{i\omega t}, \tag{12}$$

where $i^2 = -1$ and $\hat{\mathbf{u}}$, $\hat{p}$ are complex-valued functions representing the transformed displacement and pressure, respectively.

Then the coupled problem consisting of (8) and (10) after transformation into the frequency domain reads: for given $\omega \in \mathbb{C}$ find a pair $(\hat{\mathbf{u}}, \hat{p})$ satisfying

$$-\omega^2 \begin{pmatrix} \frac{1}{c_0^2} m^a\,(\hat{p},\eta) & -\rho^a\, c^s\,(\hat{\mathbf{u}},\eta) \\ 0 & \rho^s\, m^s\,(\hat{\mathbf{u}},\boldsymbol{\psi}) \end{pmatrix} + \begin{pmatrix} k^a(\hat{p},\eta) & 0 \\ c^a(p,\boldsymbol{\psi}) & k^s(\hat{\mathbf{u}},\boldsymbol{\psi}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{13}$$

for all test functions $\boldsymbol{\psi} \in \mathbf{V}$ and $\eta \in Y$.

3.3 Finite element discretization

In the case of elastic problem let us consider a finite dimensional FE space $\mathbf{V}_h \subset \mathbf{V}$, where the FE approximation $\hat{\mathbf{u}}_h$ is expressed as a linear combination of basis functions $\boldsymbol{\psi}_i \in \mathbf{V}_h$, such that $\hat{\mathbf{u}}_h = \sum_{i=1}^{2N_h} \widehat{\alpha_i^s}(\omega)\boldsymbol{\psi}_i(x)$. Similarly, the acoustic pressure can be approximated as $\hat{p}_h = \sum_{i=1}^{M_h} \widehat{\alpha_i^a}(\omega)$, where $\eta_i(x) \in Y_h \subset Y$. Substituting these approximations into (13) for a given ω results in a linear system of algebraic equations

$$\left(-\omega^2 \begin{pmatrix} \frac{1}{c_0^2}\mathbb{M}^a & -\rho^a\mathbb{C}^s \\ 0 & \rho^s\mathbb{M}^s \end{pmatrix} + \begin{pmatrix} \mathbb{K}^a & 0 \\ \mathbb{C}^a & \mathbb{K}^s \end{pmatrix}\right) \begin{pmatrix} \widehat{\boldsymbol{\alpha}^a} \\ \widehat{\boldsymbol{\alpha}^s} \end{pmatrix} = \begin{pmatrix} \mathbf{0}^a \\ \mathbf{0}^s \end{pmatrix}. \tag{14}$$

for unknown vector $\widehat{\boldsymbol{\alpha}}^T = \left(\widehat{\boldsymbol{\alpha}^a}, \widehat{\boldsymbol{\alpha}^s}\right)^T$ and the definitions of mass $\mathbb{M}^{a,s}$, coupling $\mathbb{C}^{a,s}$ and stiffness $\mathbb{K}^{a,s}$ matrices are derived from the corresponding bilinear forms (9) and (11), see also [11]. The resulting system is unfortunately non-symmetric. Nevertheless, system (14) can be solved either as an eigenvalue problem, as demonstrated in [11], or through the transfer function approach, which is the preferred method here and is described in Section 4.2, see [7].

For practical computation the Lagrange finite elements of first order are applied. The acoustic and the structure meshes are chosen to be consistent across the interface Γ_{W}, eliminating the need for additional treatment during computation of the coupling matrices.

4 Numerical results

First, a simple vibroacoustic model problem is analyzed. Next, the resonant frequencies of a coupled vibroacoustic model inspired by human phonation are determined. In all cases, the parameters are set as $\rho^f = 1.2\,\mathrm{kg/m^3}$ and $c_0 = 353\,\mathrm{m/s}$.

4.1 Simplified vibroacoustic model problem

To illustrate the basic properties of the vibroacoustic problem, as discussed in [9], we consider a simple oscillator with mass m and stiffness k coupled to an acoustic domain represented by two tubes with the following dimensions: $L_1 = 16.2\,\mathrm{cm}$, $d_1 = 2.12\,\mathrm{cm}$ and $L_2 = 26.4\,\mathrm{cm}$, $d_2 = 0.067\,\mathrm{cm}$, see Figure 2. We fix the basic circular frequency of the oscillator at $\omega_O = \sqrt{\frac{k}{m}} := 2\pi 15\,\mathrm{rad/s}$, as in [9], and we vary the oscillator mass within the range of $0.01 - 1000\,\mathrm{g}$.

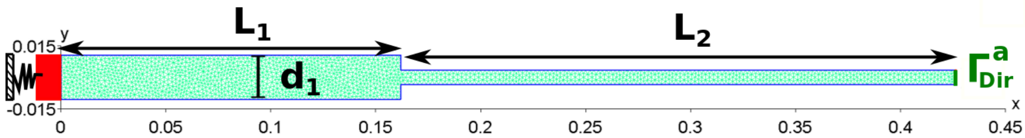

Figure 2: Simplified vibroacoustic problem consisting of (red) structure (mass on spring) and acoustic domain Ω^a composed of two tubes.

The evolution of the three lowest eigenfrequencies obtained by modal analysis as a function of mass m is shown in Figure 3. The first eigenfrequency of the system is primarily determined by the behavior of the mechanical oscillator and its value approaches the oscillator eigenfrequency as the mass m increases. The second and third natural frequencies correspond to the first two natural frequencies of the acoustic field. As the oscillator mass m increases, these frequencies asymptotically approach the first two natural frequencies of the purely acoustic problem $f_1^a = 138.0\,\text{Hz}$ and $f_2^a = 644.9\,\text{Hz}$.

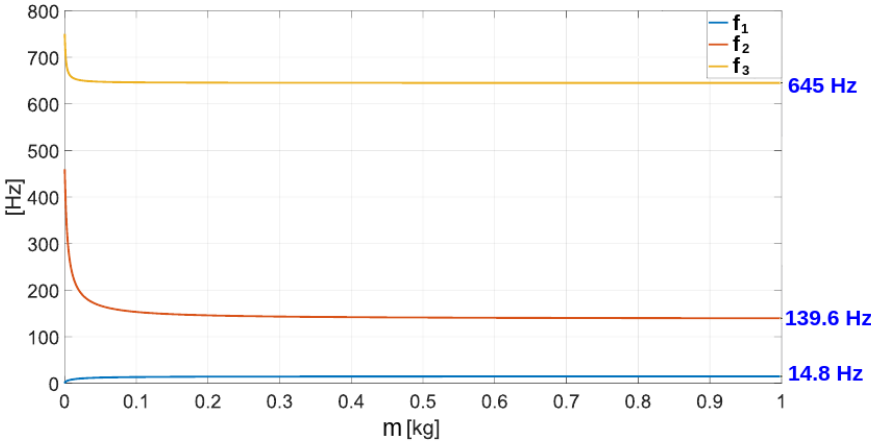

Figure 3: First three lowest eigenfrequencies of simplified coupled problem, shown as a function of the oscillator mass. The values on the right represent the frequencies calculated for $m = 1\,\text{kg}$.

The first two dominantly acoustic eigenmodes for different masses m are shown in Figure 4. It is obvious that for small masses the interface behaves as an acoustic open end, resembling a zero Dirichlet boundary condition. For large masses the interface acts acoustically as a closed end, similar to a zero Neumann boundary condition. These findings coincide well with results of [9], although the final frequency values differ. This discrepancy can be attributed to the differences between the 1D and 2D acoustic cavity models.

4.2 Resonant frequencies of vibroacoustic problem

Let us consider a simplified 2D model of a real human vocal tract extended by a narrow tube inserted into the mouth – see Fig. 5, commonly used in voice rehabilitation therapy, see [9]. The material properties and geometry of the vocal folds are chosen based on [11], with Young's modulus of elasticity $E = 8000$ N/m^2, Poisson's number $\nu = 0.4$, and density $\rho^s = 1000$ kg/m^3.

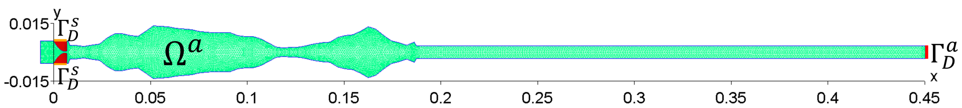

Figure 5: The model of vocal folds coupled to vocal tract of length $L = 0.1931\,\text{m}$ extended by straight tube of the same length L_2 and diameter d_2 as considered in paragraph 4.1.

We use numerical calculation of the frequency transfer function $F(\omega)$ for the determination of the natural frequencies of the coupled vibroacoustic system. This involves solving the system (14) for the response to a right-hand side excitation $\mathbf{b}$ at specified circular frequencies within the interval $\langle \omega_a, \omega_b \rangle$, as outlined in [7]. For this analysis, we set $\omega_a = 2\pi \cdot 50\,\text{rad/s}$ and $\omega_a = 2\pi \cdot 630\,\text{rad/s}$, discretized with a step size $d\omega = 0.5\,\text{rad/s}$. The excitation consists of a unit acoustic point source at $[0, 0]$ together with elastic excitations at points $[0.006, -0.001]$ and $[0.006, 0.001]\,\text{m}$. The resulting frequency transfer function $F(\omega) = \sqrt{\widehat{\boldsymbol{\alpha}}^T \mathbb{M} \widehat{\boldsymbol{\alpha}}}$ is shown in Figure 6. This function approximates

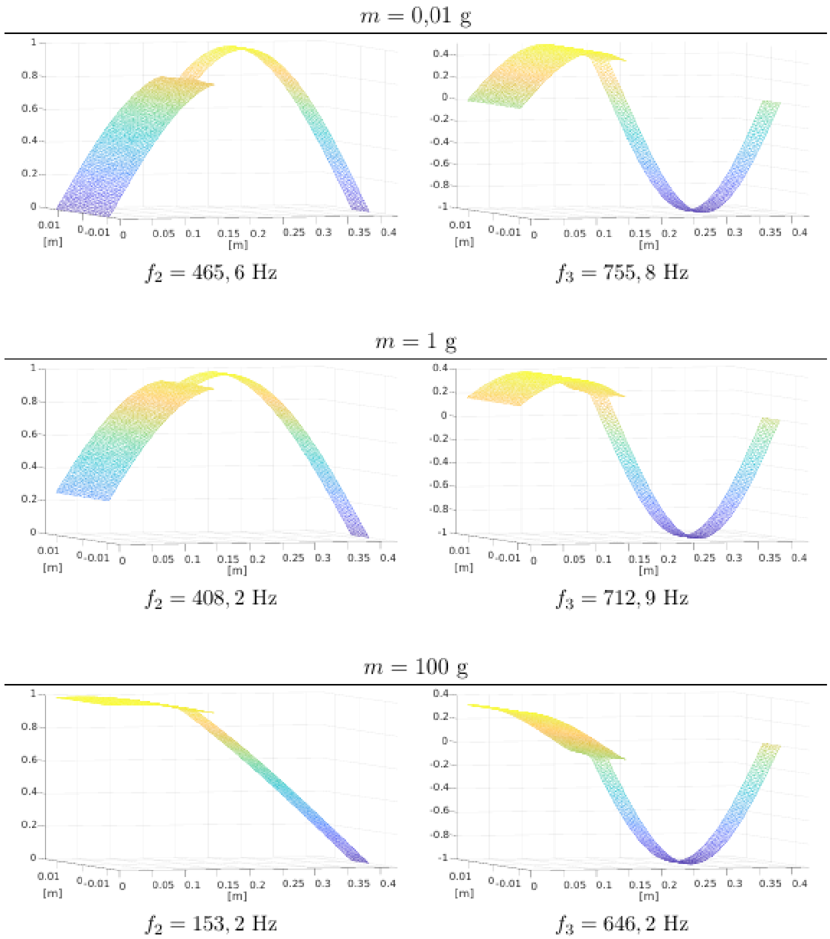

Figure 4: Eigenmodes corresponding to the acoustic frequencies f_2 and f_3 of the simplified coupled problem for varying oscillator mass. The x- and y-axes represent the geometric dimensions of the considered domain (refer to Fig. 2), while the normalized pressure is shown on the vertical z-axis and highlighted by a color scale ranging from -1 to 1.

the L2-norm of the system response $\widehat{\boldsymbol{\alpha}}$, providing a measure of the system's sensitivity to excitation at a given frequency ω.

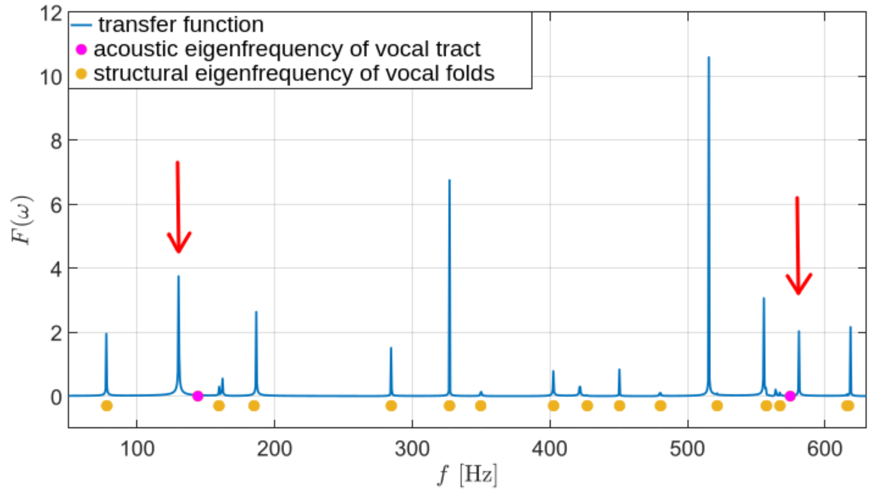

Figure 6: Computed transfer function of the coupled problem in dependence on the circular frequency, which is here converted to frequency. The natural frequencies of the pure elastic problem are highlighted by yellow dots. The first two natural frequencies 144.1 Hz and 574.7 Hz of the pure acoustic problem (purple) are shown. The red arrows indicate the resonant frequencies of the coupled problem with dominant acoustic component.

The analysis reveals that the first frequency of the coupled problem with dominant acoustic component is lower (130 Hz) than the first frequency of the pure acoustic problem (144 Hz), as observed in laboratory measurements, see [6]. In contrast, the second acoustically dominant frequency of the coupled problem is higher (581 Hz) than the second frequency of the pure acoustic problem (574.7 Hz). The acoustic components of the first two coupled eigenmodes are shown in Figure 7.

Note: Modes with dominant acoustic components are identified by their acoustic pressure amplitudes, which are several orders of magnitude larger than those in structure-dominated modes, see also [4].

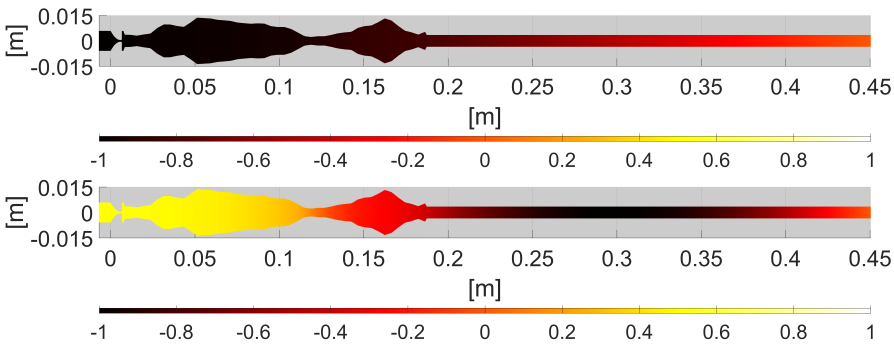

Figure 7: The first two eigenmodes of the coupled problem with dominant acoustic component at frequencies 130.1 Hz and 581.0 Hz. The normalized acoustic pressure is depicted, with the corresponding colorbar shown beneath the graphs.

5 Conclusion

The mathematical models of the acoustic and elastic subproblems, as components of the coupled vibroacoustic problem, are described in the time as well as in the frequency domain. Due to the nature of the problem, a one-way coupling is insufficient, and both couplings between the elastic body and acoustics are necessary to be considered. The vibroacoustic problem is reformulated in its weak form in order to be approximated by the finite element method. Finally, the transfer function approach is used to determinate the resonant frequencies of the coupled system and identify modes with a dominant acoustic component.

The behaviour of two vibroacoustic problems is studied. The resonant frequencies of 1D oscillator coupled with an acoustic space and their dependence on the oscillator mass agree well with the analytical description published in [9]. For the simplified 2D model of a human vocal tract extended by a narrow tube and coupled with compliant vocal cords, a 10% decrease in the frequency of the acoustically dominant eigenmode compared to the pure acoustic eigenmode is observed. However, this decrease is less pronounced than the shift reported in [9]. We hypothesize that stronger vibroacoustic coupling, leading to a more significant frequency shift, could be achieved by reducing the mass of the oscillating elastic part.

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References

[1] Brdička, M., Samek, L., Sopko, B.: *Continuum mechanics.* Academia, (2000).

[2] Davis, R. B.: Techniques to assess acoustic-structure interaction in liquid rocket engines. *PhD thesis, Duke University*, (2008).

[3] Fahy, F. J.: *Sound and structural vibration: radiation, transmission and response.* Elsevier, (2007).

[4] Gorman, D. G., Horáček, J.: Analysis of the free vibration of a coupled plate/fluid interacting system and interpretation using sub-system modal energy. *Engineering Structures*, vol. 29, no. 5: (2007) pp. 754–762.

[5] Horáček, J., Zolotarev, I.: Influence of the acoustic-structural couplings upon free vibrations of mechanical systems. *Archives of Acoustics*, vol. 18, no. 2: (1993) pp. 297–309.

[6] Horáček, J., Radolf, V., Laukkanen, A.-M.: Experimental and computational modeling of the effects of voice therapy using tubes. *Journal of Speech, Language, and Hearing Research*, (2019) pp. 1–18.

[7] Kaltenbacher, M.: *Numerical simulation of mechatronic sensors and actuators: Finite elements for computational multiphysics.* Springer, (2015).

[8] Kraxberger, F., Näger, Ch., Laudato, M., Sundström, E., Becker, S., Mihaescu, M., Kniesburges, S., Schoder, S.: On the alignment of acoustic and coupled mechanic-acoustic eigenmodes in phonation by supraglottal duct variations. *Bioengineering*, vol. 10: (2023) pp. 1369.

[9] Radolf, V., Horáček, J., Dlask, P., Otčen'ašek, Z., Geneid, A., Laukkanen, A.-M.: Measurement and mathematical simulation of acoustic characteristics of an artificially lengthened vocal tract. *Journal of Sound and Vibration*, vol. 366: (2016) pp. 556–570.

[10] Valášek, J., Sváček, P., Horáček, J.: On suitable inlet boundary conditions for fluid-structure interaction problems in a channel. *Applications of Mathematics*, vol. 64, no. 2: (2019) pp. 225–251.

[11] Valášek, J., Sváček, P., Horáček, J.: Finite element approximation of eigenvibration of a coupled vibro-acoustic system motivated by phonation into tubes. In Yilmaz, F., et al. (eds.), *Mathematical Methods for Engineering Applications*, Springer, (2022) pp. 199–207.