DOI: https://doi.org/10.14311/TPFM.2025.036

AN APPROXIMATION PROCEDURE FOR STEADY NAVIER - STOKES FLOW

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Abstract

The motion of a steady viscous incompressible fluid flow in bounded domains with a smooth boundary can be described by the nonlinear Navier-Stokes system (N). This description corresponds to the so-called Eulerian approach. We present an approximation method for (N) in the steady case by a suitable coupling of the Eulerian and the Lagrangian representation of the flow, where the latter is defined by the trajectories of the particles of the fluid. The method leads to a sequence of uniquely determined approximate solutions with a high degree of regularity, which contains a convergent subsequence with limit function v such that v is a weak solution on (N).

Keywords: Steady Navier-Stokes flow, Lagrangian approximation

1 Introduction

For the description of steady fluid flow there are in principle two approaches, the Eulerian approach and the Lagrangian approach. The first one describes the flow by its velocity $v = (v_1(x), v_2(x), v_3(x)) = v(x)$ in every point $x = (x_1, x_2, x_3)$ of the domain G containing the fluid. The second one uses the trajectory $x = (x_1(t), x_2(t), x_3(t)) = x(t) = X(t, 0, x_0)$ of a single particle of fluid, which at initial time $t = 0$ is located at some point $x_0 \in G$. The second approach is of great importance for the numerical analysis and computation of fluid flow also involving different media with interfaces [2, 3, 5, 8], while the first one has also often been used in connection with theoretical questions [4, 6, 7, 9].

It is the aim of the present note to present an approximation method for the nonlinear steady Navier-Stokes equations by coupling both the Lagrangian and the Eulerian approach. The method avoids fixpoint considerations and leads to a sequence of approximate systems, whose solution is unique and has a high degree of regularity, important at least for numerical purposes. Moreover, we show that the method allows the construction of weak solutions of the Navier-Stokes equations: The sequence of approximate solutions has at least one accumulation point satisfying the Navier-Stokes equations in a weak sense [6].

2 The Navier - Stokes Equations

We consider the steady motion of a viscous incompressible fluid in a bounded domain $G \subset \mathbb{R}^3$ with a sufficiently smooth boundary S. Because for steady flow the streamlines and the trajectories of the fluid particles coincide, both approaches mentioned above are correlated by the autonomous system of characteristic ordinary differential equations

$$x'(t) = v(x(t)), \quad x(0) = x_0 \in G, \tag{1}$$

which is an initial value problem for $t \longrightarrow x(t) = X(t, 0, x_0) = X(t, x_0)$ if the velocity field $x \longrightarrow v(x)$ is known in G.

To determine the velocity, in the present case we have to solve the steady-state nonlinear equations

$$\begin{aligned} -\nu\Delta v + v \cdot \nabla v + \nabla p = F \quad \text{in} \quad G, \\ \operatorname{div} v = 0 \quad \text{in} \quad G, \qquad v = 0 \quad \text{on} \quad S \end{aligned} \tag{2}$$

of Navier-Stokes. Here $x \longrightarrow p(x)$ is an unknown kinematic pressure function. The constant $\nu > 0$ (kinematic viscosity) and the external force density F are given data. The incompressibility of the fluid is expressed by $\operatorname{div} v = 0$, and on the boundary S we require the no-slip condition $v = 0$.

3 The Lagrangian Approach

Let us start recalling some facts, which concern existence and uniqueness for the solution of the initial value problem (1): If the function v belongs to the space $C_0^{\text{lip}}(\overline{G})$ of vector fields being Lipschitz continuous in the closure $\overline{G} = G \cup S$ and vanishing on the boundary S, then for all $x_0 \in G$ the solution

$$t \longrightarrow x(t) = X(t, x_0)$$

is uniquely determined and exists for all $t \in \mathbb{R}$ (because $v = 0$ on the boundary S, the trajectories remain in G for all times). Due to the uniqueness, the set of mappings

$$\Re = \{X(t, \cdot) : G \to G |\, t \in \mathbb{R}\}$$

defines a commutative group of C^1- diffeomorphisms on G. In particular, for $t \in \mathbb{R}$ the inverse mapping $X(t, \cdot)^{-1}$ of $X(t, \cdot)$ is given by $X(-t, \cdot)$, i.e.

$$X(t, \cdot) \circ X(-t, \cdot) = X(t, X(-t, \cdot)) = X(t - t, \cdot) = X(0, \cdot) = \text{id},$$

or, equivalently, $X(t, X(-t, x)) = x$ for all $x \in G$. It follows $\det \nabla X(t, x) = 1$ if

$$v \in C_{0,\sigma}^{\text{lip}}(\overline{G}) = \{u \in C_0^{\text{lip}}(\overline{G}) |\, \operatorname{div} u = 0\},$$

in addition. This important measure preserving property implies

$$\langle f, g\rangle = \langle f \circ X(t, \cdot), g \circ X(t, \cdot)\rangle$$

for all functions $f, g \in L^2(G)$, where $\langle \cdot, \cdot\rangle$ denotes the scalar product in $L^2(G)$.

4 The Eulerian Approach

Next let us consider the Navier-Stokes boundary value problem (2). It is well known that, given $F \in L^2(G)$, there is at least one function v satisfying (2) in some weak sense [6]. To define such a weak solution we need the space $V(G)$, being the closure of $C_{0,\sigma}^{\infty}(G)$ (smooth divergence free vector functions with compact support in G) with respect to the Dirichlet-norm $\|\nabla u\| = \sqrt{\langle \nabla u, \nabla u\rangle}$, where we define

$$\langle \nabla u, \nabla v\rangle = \sum_{i,j=1}^{3} \langle D_j u_i, D_j v_i\rangle.$$

Let us recall the following

Definition *Let $F \in L^2(G)$ be given. A function $v \in V(G)$ satisfying for all $\Phi \in C_{0,\sigma}^{\infty}(G)$ the identity*

$$\nu\langle \nabla v, \nabla \Phi\rangle - \langle v \cdot \nabla \Phi, v\rangle = \langle F, \Phi\rangle \tag{3}$$

is called a weak solution of the Navier-Stokes equations (2), and (3) is called the weak form of (2).

For a suitable approximation of the nonlinear term let us keep in mind its physical deduction. It is a convective term arising from the total or substantial derivative of the velocity vector v. Thus it seems to be reasonable to use a total difference quotient for its approximation.

To do so, let $v \in C_{0,\sigma}^{lip}(\overline{G})$ be given. Then for any $\varepsilon \in \mathbb{R}$ the mapping $X(\varepsilon, \cdot) : G \to G$ and its inverse $X(-\varepsilon, \cdot)$ are well defined. Consider for some $u \in C^1(G)$ ($C^m(G)$ is the space of continuous functions having continuous partial derivatives up to and including order $m \in \mathbb{N}$ in G) and $x \in G$ the one-sided Lagrangian difference quotients

$$\begin{aligned} L_+^\varepsilon u(x) &= \frac{1}{\varepsilon}\left[u(X(\varepsilon,\cdot)) - u(x)\right], \\ L_-^\varepsilon u(x) &= \frac{1}{\varepsilon}\left[u(x) - u(X(-\varepsilon,\cdot))\right], \end{aligned}$$

and the central Lagrangian difference quotient

$$L^\varepsilon u(x) = \frac{1}{2}\left(L_+^\varepsilon u(x) + L_-^\varepsilon u(x)\right). \tag{4}$$

Since for sufficiently regular functions

$$L_-^\varepsilon u(x) \longrightarrow v(x)\cdot\nabla u(x)$$

and

$$L_+^\varepsilon u(x) \longrightarrow v(x)\cdot\nabla u(x)$$

as $\varepsilon \to 0$, the above quotients can all be used for the approximation of the convective term $v\cdot\nabla v$. There is, however, an important advantage of the central quotient (4) with respect to the conservation of the energy:

Let $v \in C_{0,\sigma}^{lip}(G)$ and $u,\, w \in L^2(G)$. Let $X(\varepsilon,\cdot)$ and $X(-\varepsilon,\cdot)$ denote the mappings constructed from the solution of (1). Then, using the measure preserving property from above, we obtain only for the central quotient the orthogonality relation

$$\langle L^\varepsilon u,\, u\rangle = 0. \tag{5}$$

5 The Approximate System

To establish an approximation procedure we assume that some approximate velocity field v^n has been found. To construct v^{n+1} we proceed as follows:

1) Construct $X^n = X(\frac{1}{n},\cdot)$ and its inverse $X^{-n} = X(-\frac{1}{n},\cdot)$ from the initial value problem

$$x'(t) = v^n(x(t)), \qquad x(0) = x_0 \in G. \tag{6}$$

2) Construct v^{n+1} and p^{n+1} from the boundary value problem

$$\begin{aligned} -\nu\Delta v^{n+1} + \frac{n}{2}[v^{n+1}\circ X^n - v^{n+1}\circ X^{-n}] + \nabla p^{n+1} &= F \quad \text{in} \quad G, \\ \operatorname{div} v^{n+1} &= 0 \quad \text{in} \quad G, \\ v^{n+1} &= 0 \quad \text{on} \quad S. \end{aligned} \tag{7}$$

Concerning the existence and uniqueness for the solution of (6) and (7) we need the usual Sobolev Hilbert spaces $H^m(G)$, $m \in \mathbb{N}$, which denote the closure of $C^m(G)$ with respect to the norm $\|\cdot\|_{H^m}$ (see [1]). Now we can formulate

Theorem a) *Assume $v^n \in H^3(G)\cap V(G)$ and $F \in H^1(G)$. Then for all $x_0 \in G$ the initial value problem (6) is uniquely solvable, and the mappings*

$$X^n : G \to G, \qquad X^{-n} : G \to G$$

are measure preserving C^1- diffeomorphisms in G. Moreover, there is a uniquely determined solution

$$v^{n+1} \in H^3(G)\cap V(G),\ \nabla p^{n+1} \in H^1(G)$$

of the equations (7). The velocity field v^{n+1} satisfies the energy equation

$$\nu\|\nabla v^{n+1}\|^2 = \langle F,\, v^{n+1}\rangle.$$

b) *Assume* $v^0 \in H^3(G) \cap V(G)$ *and* $F \in H^1(G)$. *Let* (v^n) *denote the sequence of solutions constructed in view of Part a). Then* (v^n) *is bounded in* $V(G)$ *i.e.* $\|\nabla v^n\|^2 \leq C_{G,F,\nu}$ *for all* $n \in \mathbb{N}$, *where the constant* $C_{G,F,\nu}$ *does not depend on* n. *Moreover,* (v^n) *has an accumulation point* $v \in V(G)$ *satisfying (3), i.e.* v *is a weak solution of the Navier-Stokes equations (2).*

The above theorem can be proved as follows: Because $H^3(G)$ is continuously imbedded in $C^1(\overline{G})$ [1] and thus $H^3(G) \cap V(G) \subset C^1_{0,\sigma}(\overline{G})$, the initial value problem (6) is uniquely solvable, and X^n as well as X^{-n} have the asserted properties.
Consider now the boundary value problem (7). By means of a Galerkin method we can prove the existence of some function $v^{n+1} \in V(G) \subset H^1(G)$ satisfying the weak version of (7). Moreover, due to the linearity of the problem there is exactly one weak solution v^{n+1}. Next we prove the regularity property $v^{n+1} \in H^3(G)$. To do so we write (7) in the form of a linear Stokes system with the right hand side

$$K = F - \frac{n}{2}\Big[v^{n+1} \circ X^n - v^{n+1} \circ X^{-n}\Big].$$

and use Cattabriga's estimate to obtain $v^{n+1} \in H^3(G)$ if only $K \in H^1(G)$. Thus it suffices to show $v^{n+1} \circ X^n \in H^1(G)$ ($v^{n+1} \circ X^{-n}$ analogously), which follows with methods of differential inequalities. The energy equation is obtained from the orthogonality relation (6). This proves Part a) of the Theorem.

The boundedness of the above constructed sequence (v^n) in $V(G)$ obviously follows from the energy equation. Because (v^n) is bounded in $V(G)$, there is a convergent subsequence - again denoted by (v^n) - with limit $v \in V(G)$ such that $v^n \longrightarrow v$ weakly with respect to the Dirichlet norm $\|\nabla \cdot\|$. Because the imbedding $V(G) \subset L^2(G)$ is compact [1] we can again extract a subsequence such that $v^n \longrightarrow v$ strongly in $L^2(G)$, in addition. Now it can be shown that theses two properties are already sufficient to proceed to the limit also in the convective term,. As a result we obtain that the limit function v constructed above is a weak solution of the Navier-Stokes equations (2).

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